Graded Matrices

Create mappings $L \longrightarrow L$ where $L = L_0 \oplus L_1$ can be represented as a graded matrix; M:

If L_0 has n generators
" L_1 " m generators

$$M = \left(\begin{array}{c|c} A_{n\times n} & B_{n\times m} \\ \hline C_{m\times n} & D_{m\times m} \end{array}\right)$$

– also called supermatrix

– acts on vector space spanned by

$$v \equiv \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \quad \text{where } v_0 \in L_0 \text{ and } v_1 \in L_1$$

$A \equiv A^{ab}$ $\qquad$ $a, b = \{1, 2, \dots n\}$

$B \equiv B^a{}_i$ $\qquad v_0 \equiv v_0{}^a$ $\qquad i, j = \{1, 2, \dots m\}$

$C \equiv C_i{}^a$ $\qquad v_1 \equiv v_{1i}$

$D \equiv D_{ij}$

Then: $$Mv = \begin{pmatrix} A^{ab} & B^a{}_j \\ C_i{}^b & D_{ij} \end{pmatrix}\begin{pmatrix} v_0{}^b \\ v_{1j} \end{pmatrix} = \begin{pmatrix} A^{ab} v_0{}^b + B^a{}_j v_{1j} \\ C_i{}^b v_0{}^b + D_{ij} v_{1j} \end{pmatrix} = \begin{pmatrix} v_0' \\ v_1' \end{pmatrix} = v'$$

$$\left.\begin{array}{l} v_0 \ \& \ v_0' \text{ is an even quantity} \\ \text{and } v_1 \ \& \ v_1' \text{ is an odd quantity} \end{array}\right\} \Rightarrow \begin{array}{l} A^{ab} \ \& \ D_{ij} \text{ are even} \\ B^a{}_j \ \& \ C_i{}^b \text{ are odd} \end{array}$$

$$\Rightarrow \boxed{\begin{array}{l} A^{ab} \ \& \ D_{ij} \text{ are c-number-valued} \\ B^a{}_j \ \& \ C_i{}^b \text{ are Grassman-valued.} \end{array}}$$

Therefore $$BC = B^a{}_j C_j{}^b = -C_j{}^b B^a{}_j \quad \text{(minus sign!)}$$
$$= -(C^b{}_j)^T (B_j{}^a)^T$$

so that $\boxed{(BC)^T = -C^T B^T,}$ with a minus sign

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Supertrace:

$$\mathrm{STr}(M) := \mathrm{Tr}\,A - \mathrm{Tr}\,D$$

Satisfies cyclic property $\mathrm{STr}[\underbrace{M_1 M_2 \ldots M_x}_{\text{string of supermatrices}}] = \mathrm{STr}[M_x M_1 M_2 \cdots]$

Superdeterminant:

$$\mathrm{Sdet}(M) := e^{\mathrm{STr}\ln M} \qquad \text{where } M \equiv \exp X$$

Property: $\mathrm{STr}\ln M = \ln \mathrm{Sdet}\, M$.

Property: $\mathrm{Sdet}(M_1 M_2 \ldots) = \mathrm{Sdet}(M_1) + \mathrm{Sdet}(M_2) + \ldots$

Can write superdeterminant in terms of ordinary determinants:

$$\mathrm{Sdet}\, M = \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)}$$

Super transpose:

$$M^{ST} = \begin{pmatrix} A^T & -C^T \\ B^T & D^T \end{pmatrix}$$

defined so as to mimic ordinary law of transpose: $(M_1 M_2)^{ST} = M_2^{ST} M_1^{ST}$

Property: $\mathrm{Sdet}(M^{ST}) = \mathrm{Sdet}(M)$