Now vector superfield D term — Fayet–Iliopoulos

For $U(1)$ vector superfield, $D(x)$ is gauge invariant
(no inhomogenous part, and homogeneous part vanishes because $f^{abc}=0$)

Then,

$$S = \ldots \int d^4x \; -2\kappa [V]_D$$

$$= \ldots \int d^4x \; (-2\kappa)\Big[\tfrac{1}{2}D - \underbrace{\tfrac{1}{4}\partial^2\phi}_{\text{total derivative.}}\Big]$$

$$= \ldots \int d^4x \; (-\kappa D)$$

"Fayet–Iliopoulos term"

[Its effect will be apparent upon eliminating $D(x)$]

Then, putting F & D terms together,

$$S = \int d^4x \Big[\underbrace{\lambda^\dagger i\bar\sigma^\mu \partial_\mu \lambda + \tfrac{1}{2}D^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{F-\text{term}} - \kappa D \Big]$$

D-term.
(only for $U(1)$ gauge theory)

To couple U(1) vector superfields V to chiral superfields Φ and Φ^*, ask how Φ and Φ^* transform:

In ordinary field theory, $\psi \to e^{i\alpha(x)}\psi$

In SUSY, it can't be $\Phi \longrightarrow e^{i\alpha(x)}\Phi$,
because R.H.S. isn't a superfield (since $\alpha(x)$ isn't a superfield).

Obviously need:

$$\begin{cases} \Phi_i \longrightarrow e^{2g i Q \Omega(x,\theta,\theta^\dagger)}\,\Phi_i \\ \Phi^{*i} \longrightarrow e^{-2g i Q \Omega^\dagger(x,\theta,\theta^\dagger)}\,\Phi^{*i} \end{cases}$$

$\Omega \equiv$ transformation parameter superfield.

Notice: this is not an element of U(1).

This implies ordinary D-term (for Φ & Φ^*) isn't gauge invariant.

$$\left[\Phi^{*i}\,\Phi_i\right]_D \xrightarrow{U(1)} \left[\Phi^{*i}\, e^{-2g i Q(\Omega^*-\Omega)}\,\Phi_i\right]_D$$

Fix by introducing vector superfield with gauge transformation rule:

$$V \longrightarrow V - i(\Omega - \Omega^*)$$

So that new (gauge-invariant) D-term is

$$\mathcal{L} = \left[\Phi^{*i}\left(e^{2gQV}\right)\Phi_i\right]_D .$$

Comparator

That this is gauge-invariant is obvious: $(\Omega - \Omega^*)$ from V cancels against that from Φ & Φ^*.

Next step: Write in terms of component fields.

Notice: In Wess-Zumino gauge, the comparator has a terminating Taylor expansion:

Formulae:

$$(V_{W.Z.})^2 = (\theta\sigma^\mu\theta^\dagger)(\theta\sigma^\nu\theta^\dagger)A_\mu A_\nu + \text{vanishing cross terms.}$$

$$= \frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger A^2(x)$$

$$(V_{W.Z})^3 = 0.$$

Comparator:

$$e^{2gQV_{W.Z}} = 1 + 2gQV_{W.Z} + \frac{1}{2!}(2gQ)^2(V_{W.Z})^2 + \mathcal{O}(V_{W.Z})^3$$

$$= 1 + 2gQ\left[\theta\sigma^\mu\theta^\dagger A_\mu + \theta^\dagger\lambda^\dagger\theta\theta + \theta^\dagger\theta^\dagger\theta\lambda + \frac{1}{2}\theta^\dagger\theta^\dagger\theta\theta D\right]$$
$$+ 2g^2Q^2\frac{1}{2}\theta\theta\theta^\dagger\theta^\dagger A_\mu A^\mu.$$

Then insert expansion into D-term:

$$\left[\Phi^{*i}\left(e^{2gQV_{W.Z.}}\right)\Phi_i\right]_D$$

the '1'-term of comparator leads to usual kinetic terms.

$$= \ldots(\theta\theta\theta^\dagger\theta^\dagger)\left[-\frac{1}{4}\phi^*\partial^2\phi - \frac{1}{2}\partial_\mu\phi^*\partial^\mu\phi - \frac{1}{4}(\partial^2\phi)^*\phi\right. \quad (*)$$
$$\left. + \frac{i}{2}\chi^\dagger\bar\sigma^\mu\partial_\mu\chi - \frac{i}{2}\partial_\mu\chi^\dagger\bar\sigma^\mu\chi + F^*F\right]$$

then each subsequent term in comparator cross multiplies from Φ^{*i} and Φ_i giving c.c.

$$= \ldots 2gQ\left[\theta\sigma^\mu\theta^\dagger A_\mu(i\theta\sigma^\nu\theta^\dagger\partial_\nu\phi^*)\phi + \text{c.c.} + \underbrace{\theta\sigma^\mu\theta^\dagger A_\mu\sqrt{2}\,\theta^\dagger\chi^\dagger\sqrt{2}\,\theta\chi}_{\text{simplify.}}\right.$$
$$+ \theta^\dagger\lambda^\dagger\theta\theta\sqrt{2}\,(\theta^\dagger\chi^\dagger)\phi + \text{c.c.}$$
$$\left. + \frac{1}{2}\theta^\dagger\theta^\dagger\theta\theta D\,\phi\phi^*\right] + g^2Q^2\theta\theta\theta^\dagger\theta^\dagger A_\mu A^\mu\phi^*\phi.$$

simplification:

$$\theta^\alpha \sigma^\nu_{\alpha\dot\beta} \theta^{\dagger\dot\beta} \chi^\dagger_{\dot\gamma} \theta^{\dagger\dot\gamma} \theta^\delta \chi_\delta$$

$$= \tfrac{1}{2}\epsilon^{\alpha\delta}(\theta\theta)\ \tfrac{1}{2}\epsilon^{\dot\gamma\dot\beta}(\theta^\dagger\theta^\dagger)\ \sigma^\mu_{\alpha\dot\beta}\ \chi^\dagger_{\dot\gamma}\chi_\delta$$

$$= \tfrac{1}{4}(\theta\theta)(\theta^\dagger\theta^\dagger)\ \sigma^\mu_{\alpha\dot\beta}(-\chi^{\dagger\dot\beta})\chi^\alpha$$

$$= +\tfrac{1}{4}\ \theta\theta\theta^\dagger\theta^\dagger\ \chi\sigma^\mu\chi^\dagger$$

$$= -\tfrac{1}{4}\ \theta\theta\theta^\dagger\theta^\dagger\ \chi^\dagger\sigma^\mu\chi$$

$$= \ldots (\theta\theta\theta^\dagger\theta^\dagger)\Big[-igQA_\mu\ \phi^*\overleftrightarrow{\partial}^\mu\phi + g^2Q^2A_\mu A^\mu\phi^*\phi + gQA_\mu A^\mu\chi^\dagger\bar\sigma^\mu\chi - \sqrt{2}\,gQ(\phi\chi^\dagger\lambda^\dagger + \phi^*\chi\lambda) + gQD\phi^*\phi\Big] \qquad (**)$$

(and identify covariant derivatives)

Thus, upon adding (*) and (**) together, the kinetic terms (D-term) is:

$$S = \int d^4x\ \left[\phi^{*i}\left(e^{2gQV_{W.Z}}\right)\phi_i\right]_D$$

$$= \int d^4x\Big[(D_\mu\phi)^* D^\mu\phi + i\chi^\dagger\bar\sigma^\mu D_\mu\chi + F^*F - \sqrt{2}\,g\,Q\,(\phi\chi^\dagger\lambda^\dagger + \phi^*\chi\lambda) + gQD\phi\phi^*\Big]$$

Q = of chiral superfield

$Q\,\Phi = q\,\Phi$.

where $D_\mu\phi = (\partial_\mu + igQA_\mu)\phi$

$D_\mu\chi = (\partial_\mu + igQA_\mu)\chi$

$D_\mu\lambda = \partial_\mu\lambda$

[Note: since D-term is gauge-invariant by construction, its form in terms of component fields would been the same in any gauge]

(fields like ϕ, χ, F in vector superfield would have canceled away).