Superfield / Superspace formulation of SUSY QM

Observe: the composition of two finite (active) SUSY transformations:

$$U(t', \alpha, \alpha^*)\, U(t, \theta, \theta^*)$$

$$= e^{i\hat{H}t'/\hbar + \alpha\hat{Q} + \alpha^*\hat{Q}^\dagger}\, e^{i\hat{H}t/\hbar + \theta\hat{Q} + \theta^*\hat{Q}^\dagger}$$

Use Baker-Campbell-Hausdorff formula to combine exponentials:

$$e^A e^B = e^{A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[B,A]] + \dots}$$

$$A = i\hat{H}t'/\hbar + \alpha\hat{Q} + \alpha^*\hat{Q}^\dagger$$
$$B = i\hat{H}t/\hbar + \theta\hat{Q} + \theta^*\hat{Q}^\dagger$$

$$[A,B] = [\alpha\hat{Q} + \alpha^*\hat{Q}^\dagger,\ \theta\hat{Q} + \theta^*\hat{Q}^\dagger]$$
$$= [\alpha\hat{Q}, \theta^*\hat{Q}^\dagger] + [\alpha^*\hat{Q}^\dagger, \theta\hat{Q}]$$

evaluate commutators: use $\{Q, Q^\dagger\} = 2H$

$$[\alpha\hat{Q}, \theta^*\hat{Q}^\dagger] = \alpha\hat{Q}\theta^*\hat{Q}^\dagger - \theta^*\hat{Q}^\dagger\alpha\hat{Q}$$
$$= -\alpha\theta^*\hat{Q}\hat{Q}^\dagger + \underbrace{\theta^*\alpha}_{\ominus}\hat{Q}^\dagger\hat{Q}$$
$$= -\alpha\theta^*\hat{Q}\hat{Q}^\dagger - \alpha\theta^*\hat{Q}^\dagger\hat{Q}$$
$$= -\alpha\theta^*\underbrace{\{\hat{Q}, \hat{Q}^\dagger\}}_{2\hat{H}}$$
$$= -\alpha\theta^*\, 2\hat{H}$$

similarly $[\alpha^*\hat{Q}^\dagger, \theta\hat{Q}] = -\alpha^*\theta\, 2\hat{H}$

$$[A,B] = -(\alpha\theta^* + \alpha^*\theta)\, 2\hat{H}$$

Since $\hat{H}$ commutes with all generators, series terminates in exponent.

$$\Rightarrow e^{A} e^{B} = e^{i\hat{H}t/\hbar + \alpha\hat{Q} + \alpha^{*}\hat{Q}^{\dagger} + i\hat{H}t'/\hbar + \theta\hat{Q} + \theta^{*}\hat{Q}^{\dagger} + \frac{1}{2}(-\alpha\theta^{*} - \alpha^{*}\theta)2H}$$

$$= e^{(i\frac{t}{\hbar} + i\frac{t'}{\hbar} - \alpha\theta^{*} - \alpha^{*}\theta)\hat{H} + (\alpha+\theta)\hat{Q} + (\alpha^{*}+\theta^{*})\hat{Q}^{\dagger}}$$

$$= e^{i(t+t' + i\hbar\alpha\theta^{*} + i\hbar\alpha^{*}\theta)\hat{H}/\hbar + (\alpha+\theta)\hat{Q} + (\alpha^{*}+\theta^{*})\hat{Q}^{\dagger}}.$$

Implies: making a series of two successive finite arbitrary SUSY transformations amounts to performing a total SUSY transformation of $\alpha+\beta$ (and $\alpha^{*}+\beta^{*}$), but also to "evolving" the system in time by an amount $i\hbar(\alpha\theta^{*} + \alpha^{*}\theta)$.

Discussion is in space of group transformations, but we can interpret this as a SUSY transformation acting on a homogeneous space with coordinates (t, θ, θ^{*}) identified by the (left) coset space spanned by the SUSY group elements $\Omega(t,\theta,\theta^{*}) = e^{i\hat{H}t/\hbar + \theta Q + \theta^{*}Q^{\dagger}}$.

Bottom line:

under a SUSY transformation,

$$\begin{pmatrix} t \\ \theta \\ \theta^{*} \end{pmatrix} \longrightarrow \begin{pmatrix} t + t' + i\hbar(\alpha\theta^{*} + \alpha^{*}\theta) \\ \theta + \alpha \\ \theta^{*} + \alpha^{*} \end{pmatrix}$$

formally implemented by:

$$\hat{U}(t',\alpha,\alpha^{*})\underbrace{\Omega(t,\theta,\theta^{*})}_{\text{coset element.}} = \Omega(t+t'+i\hbar(\alpha\theta^{*}+\alpha^{*}\theta),\ \theta+\alpha,\ \theta^{*}+\alpha^{*})$$

This is our superspace