Problems 3

1. Consider a supersymmetric mechanical system of one superfield $X(t, \theta, \theta^*)$, interacting through the superpotential,

$$W(x) = C + \tfrac{1}{2} m\omega x^2 + \tfrac{1}{6} y x^3,$$

with ω, $y > 0$. Write down the supersymmetric Lagrangian for the dynamical coordinates x (describing a hypothetical particle), and ψ and ψ^* (abstract "Fermion particle").

For simplicity, set $y = 0$ for now. Then repeat exercise with $y \neq 0$.

a) What is the potential $V(x)$ felt by the particle at x? make a sketch.

b) Falling back on the Pauli spin interpretation of ψ and ψ^*, what is the magnetic induction (in the $\hat{3}$ direction) as a function of x felt by the spin? make a sketch.

c) *Small oscillation analysis*

(i) Find the equilibrium stable points of the potential $V(x)$, and determine the small oscillation frequency about that point.

(ii) Suppose the spin of this particle is not aligned along the $\hat{3}$-direction. The spin will undergo Larmor precession. At the equilibrium points of the potential $V(x)$, determine the precession frequency for the spin about the $\hat{3}$ axes.

(iii) Compare to part (i), and admire the magic of supersymmetry! Qualitatively describe the simultaneous motion of the particle and its spin in the magnetic field.

2. *Mind-bending dimensional analysis-paradox.*

a) Convince yourself that the Grassmann coordinate θ and θ^* carry dimensions $T/(LM^{1/2}) = (\text{energy})^{-1/2}$. Hint: use the supersymmetry algebra, and the unitary exponential form for a general supersymmetry transformation.

b) Write down the expansion for a general superfield $X(t, \theta\,\theta^*)$ in terms of its component functions.

(i) Compute the derivative $\frac{\partial}{\partial\theta} X$, and determine the dimensions of the result.

(ii) Do the same for the integral $\int d\theta X$.

c) Does this make sense? Resolve paradoxes if any.

Solutions

1. $$W(x) = C + \frac{1}{2} m\omega x^2 + \frac{1}{6} y x^3$$

$$S = \int dt \left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2m} (W')^2 + i \psi^* \dot{\psi} - \frac{1}{2m} [\psi^*, \psi] W''(x) \right)$$

$$W'(x) = m\omega x + \frac{1}{2} y x^2 \quad \Rightarrow \quad (W')^2 = m^2\omega^2 x^2 + m\omega y x^3 + \frac{1}{4} y^2 x^4$$

$$\cancel{W''(x) = m\omega + yx}$$

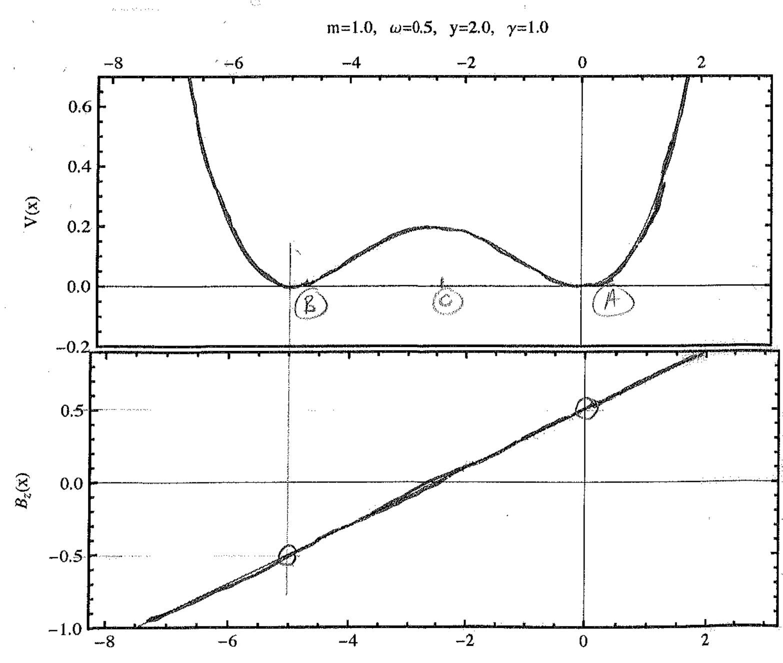
$$\Rightarrow \quad L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m\omega^2 x^2 - \frac{1}{2} \omega y x^3 - \frac{y^2}{8m} x^4$$
$$+ i \psi^* \dot{\psi} - \omega \psi^* \psi - \frac{1}{m} y \psi^* \psi x$$

a) $$V(x) = \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} \omega y x^3 + \frac{y^2}{8m} x^4$$

(sketch below)

b) $$B_z(x) = \frac{1}{\gamma m} W''(x)$$ γ = magnetomechanical ratio $\left(\frac{-eg}{2m}\right)$

$$= \frac{\omega}{\gamma} + \frac{yx}{\gamma m}$$



c) (i) Equilibrium stable points:

Critical points: $\frac{\partial V}{\partial x} = m\omega^2 x + \frac{3}{2}\omega\gamma x + \frac{1}{2}\frac{\gamma^2}{m}x^2 = 0$

solutions: $x = \left\{ \overset{Ⓐ}{0}, \ \overset{Ⓑ}{\frac{-2m\omega}{\gamma}}, \ \overset{Ⓒ}{\frac{-m\omega}{\gamma}} \right\}$

$\frac{\partial^2 V}{\partial x^2} = \left\{ m\omega^2, \ m\omega^2, \ \frac{-m\omega^2}{2} \right\}$

stable points

Oscillation frequencies:

$x_A = 0 \Rightarrow \boxed{\omega_A = \omega}$

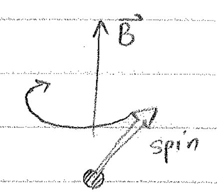
$x_B = \frac{-2m\omega}{\gamma} \Rightarrow \boxed{\omega_B = \omega}$

(ii) Larmor precession:

$\vec{\omega}_L = -\gamma \vec{B}$

at point A:

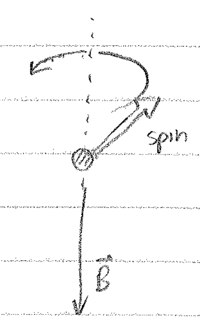
$\vec{B}(x_A) = \frac{\omega}{\gamma} \Rightarrow \boxed{\omega_L = -\omega}$



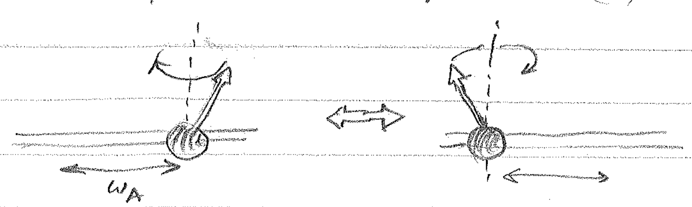
at point B:

$\vec{B}(x_B) = \frac{\omega}{\gamma} + \frac{\gamma}{\gamma m}\left(\frac{-2m\omega}{\gamma}\right)$

$= \frac{-\omega}{\gamma} \Rightarrow \boxed{\omega_L = +\omega}$



(iii) oscillation frequency ω_A, ω_B equal to Larmor precession frequency ω_L (!)



oscillation completes cycle in same time as precession completes cycle.

$\rightarrow$ synchronized