Matrix representation for Grassmann generators

for $\underline{N=1}$ (just one generator),

$$\lambda = z_0 + z_1 \zeta^1 \qquad \mathrm{rep}(\zeta^1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for $\underline{N=2}$

$$\lambda = z_0 + z_1 \zeta^1 + z_2 \zeta^2 + z_{12} \zeta^1 \zeta^2$$

$$\mathrm{rep}(\zeta^1) = \left(\begin{array}{cc|cc} 0 & 0 & & \\ 1 & 0 & & \\ \hline & & 0 & 0 \\ & & 1 & 0 \end{array}\right) \qquad \mathrm{rep}(\zeta^2) = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline 1 & & & \\ & -1 & & \end{array}\right) \qquad \mathrm{rep}(\zeta^1\zeta^2) = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline 0 & 0 & & \\ 1 & 0 & & \end{array}\right)$$

$$\mathrm{rep}(\zeta^2\zeta^1) = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline 0 & 0 & & \\ -1 & 0 & & \end{array}\right)$$

for general N,
need $2^N \times 2^N$ matrices.

Complex conjugation of supernumbers

The conjugation operator for complex numbers is defined: $\quad z = a + ib \Rightarrow z^* = a - ib$

For supernumbers, $\lambda \in \Lambda_N$

$$\lambda = z_0 + z_i \zeta^i + \tfrac{1}{2} z_{ij} \zeta^i \zeta^j + \ldots$$

$$\Rightarrow \lambda^* = z_0^* + z_i^* (\zeta^i)^* + \tfrac{1}{2} z_{ij}^* (\zeta^i \zeta^j)^* + \ldots$$

we still need to define conjugation for Grassmann generators

<u>convention</u>:

$$(\zeta^i)^* = \zeta^i \qquad \text{① reality condition}$$
$$(\zeta^i \zeta^j)^* = \zeta^j \zeta^i \qquad \text{② reverse order of multiplication.}$$
$$(\zeta^i \zeta^j \zeta^k)^* = \zeta^k \zeta^j \zeta^i \qquad \text{(no minus sign)}$$
$$\vdots \qquad \vdots \qquad \vdots$$

Reality condition, together with reverse ordering implies:

$$(\zeta^i)^* = \zeta^i$$
$$(\zeta^i\zeta^j)^* = \zeta^j\zeta^i = -\zeta^i\zeta^j$$
$$(\zeta^i\zeta^j\zeta^k)^* = \zeta^k\zeta^j\zeta^i = +\zeta^i\zeta^j\zeta^k$$
$$\vdots$$

A supernumber $\lambda \in \Lambda_N$ is said to be

- purely real if $\lambda^* = \lambda$ $\Rightarrow$ happens if z-even coefficients $\in \mathbb{R}e$. and z-odd coefficients $\in \mathbb{I}m$.
- purely imaginary if $\lambda^* = -\lambda$ $\Rightarrow$ " if z-even coefficients $\in \mathbb{I}m$ and z-odd coefficients $\in \mathbb{R}e$.

n.b. just because a supernumber is real doesn't mean it is real-valued (i.e. in $\mathbb{R}$), for example the QED Lagrangian: $\mathcal{L} = \bar{\psi}(i\not{D}-m)\psi - \frac{1}{4}F^2$ is real but not real-valued.

In physics, it is useful to express supernumbers in terms of complex Grassmann generators.

Construction: (obviously) require Λ_N with N even.

consider Λ_2: $\{\zeta^1, \zeta^2\}$.

then
$$\left.\begin{aligned}\xi &= \frac{1}{\sqrt{2}}(\zeta^1 + i\zeta^2)\\ \xi^* &= \frac{1}{\sqrt{2}}(\zeta^1 - i\zeta^2)\end{aligned}\right\} \qquad \begin{aligned}\zeta^1 &= \frac{1}{\sqrt{2}}(\xi + \xi^*)\\ \zeta^2 &= \frac{-i}{\sqrt{2}}(\xi - \xi^*)\end{aligned}$$

a supernumber $\lambda \in \Lambda_2$ may be expressed:

$$\lambda = z_0 + z_1\zeta^1 + z_2\zeta^2 + z_{12}\zeta^1\zeta^2$$

$$= z_0 + z_1\frac{1}{\sqrt{2}}(\xi+\xi^*) + z_2\frac{-i}{\sqrt{2}}(\xi - \xi^*) + z_{12}\frac{-i}{2}(\xi+\xi^*)(\xi-\xi^*)$$

organize by ξ, ξ^*

$$\lambda = z_0 + \underbrace{\frac{1}{\sqrt{2}}(z_1 - i z_2)}_{z'^*}\xi + \underbrace{\frac{1}{\sqrt{2}}(z_1 + i z_2)}_{z'}\xi^* - \frac{i z_{12}}{2}\left(-\xi\xi^* + \xi^*\xi\right)$$

$$= z_0 + \underbrace{z'^*\xi + z'\xi^*}_{\text{this is the Fermionic (Grassmann-odd) part}} + i z_{12}\xi^*\xi$$

used to describe Fermionic fields.

- generator at each space & time point.

alternatively, generator for each Fourier mode.