Calculus of Grassmann (odd) variables

<u>Differentials</u> involving Grassmann (odd) variables $\theta_1, \theta_2, \ldots \in \Lambda_N^a$ are defined by:

(*) $$d(\theta_1 \theta_2 \cdots \theta_n) = \sum_{i=1}^{n} \theta_1 \theta_2 \cdots (d\theta_i) \cdots \theta_n$$ (product rule) [Leibniz rule]

The derivative acting on a differential is defined by

(**) $$\left(\frac{d}{d\theta_j}\, d\theta_i\right)(\theta_1 \theta_2 \cdots \theta_n) = \delta_i{}^j \,(\theta_1 \theta_2 \cdots \theta_n)$$

Difference is in the positioning of the differential
- important when complex Grassmann-odd variables are involved.

<u>Left derivative</u>: push $d\theta$ to left

$$d(\theta_1 \theta_2 \cdots \theta_n) = \sum_{i=1}^{n} (-1)^{i+1} (d\theta_i)\,(\theta_1 \theta_2 \cdots [\substack{\text{missing} \\ \theta_i}] \cdots \theta_n)$$

then

$$\frac{\vec{d}}{d\theta_j}(\theta_1 \theta_2 \cdots \theta_n) = \sum_{i=1}^{n} (-1)^{i+1} \left(\frac{d}{d\theta_j}\, d\theta_i\right)(\theta_1 \theta_2 \cdots [\substack{\text{missing} \\ \theta_i}] \cdots \theta_n)$$

$$= (-1)^{j+1} (\theta_1 \theta_2 \cdots [\substack{\text{missing} \\ \theta_j}] \cdots \theta_n)$$

<u>Right derivative</u>: push $d\theta$ to right

$$d(\theta_1 \theta_2 \cdots \theta_n) = \sum_{i=1}^{n} (-1)^{n-j} (\theta_1 \theta_2 \cdots [\substack{\text{missing} \\ \theta_i}] \cdots \theta_n)\, d\theta_i$$

then

$$(\theta_1 \theta_2 \cdots \theta_n) \frac{\overleftarrow{d}}{d\theta_j} = \sum_{i=1}^{n} (-1)^{n-j} (\theta_1 \theta_2 \cdots [\substack{\text{missing} \\ \theta_i}] \cdots \theta_n) \left(\frac{d}{d\theta_j}\, d\theta_i\right)$$

$$= (-1)^{n-j} (\theta_1 \theta_2 \cdots [\substack{\text{missing} \\ \theta_j}] \cdots \theta_n)$$

Integration over Grassmann-odd variables, written $\int d\theta\, f(\theta)$
over space of (analytic) superfunctions $f(\theta) = c_0 + c_1\theta + \ldots$
are required to satisfy:

① Linearity: $\int d\theta \left(f(\theta) + g(\theta)\right) = \int d\theta\, f(\theta) + \int d\theta\, g(\theta)$

② Shift-invariant: $\int d\theta\, f(\theta + \theta') = \int d\theta\, f(\theta)$

(like integration over whole $\mathbb{R}$)

③ Invariance under change of variables: $\int d\theta\, f(\theta) = \int d\tilde{\theta}\, f(\tilde{\theta})$

Result: Integration follows rules of "Berezin integrals" up to normalization.

$$\int d\theta\, f(\theta) = \mathcal{N} \frac{\overrightarrow{d}}{d\theta} f(\theta)$$ left integral = left derivative

$$\int f(\theta)\, d\theta = \mathcal{N}\, f(\theta) \frac{\overleftarrow{d}}{d\theta}$$ Right integral = right derivative.

④ Normalization: Kleinert uses $\mathcal{N} = \sqrt{2\pi}$

Everybody else $\mathcal{N} = 1$

The Jacobian for Grassmann Integration (real θs)

If $P(\theta) = c_0 + c_1\theta$, then

$$\int d\theta \; P(\theta) = c_1 \qquad (*)$$

Consider a linear change of variables

$$\theta \longrightarrow \tilde{\theta}(\theta) = a\theta + b \qquad a, b \in \Lambda^c$$

expect

$$\int d\tilde{\theta} \; P(\tilde{\theta}(\theta)) = c_1 \qquad (**)$$

Start with $(*)$, and insert change of variables.

$$\int d\theta \; P(\tilde{\theta}(\theta)) = \int d\theta \left[c_0 + c_1(a\theta + b) \right] \overset{!}{=} c_1 a$$

identify $d\theta/d\tilde{\theta}$

Divide both sides by $a = d\tilde{\theta}/d\theta$

$$\left(\frac{d\tilde{\theta}}{d\theta}\right)^{-1} \int d\theta \; P(\tilde{\theta}(\theta)) = \int d\theta \left(\frac{d\tilde{\theta}}{d\theta}\right)^{-1} P(\tilde{\theta}(\theta)) = c_1 = \int d\tilde{\theta} \; P(\tilde{\theta})$$

indep of θ, because linear transformation. match against $(**)$

$$\Rightarrow \boxed{d\tilde{\theta} = \left(\frac{d\tilde{\theta}}{d\theta}\right)^{-1} d\theta}$$

Generalization to more complicated case with n variables

$$P(\theta_1, \theta_2, \ldots) = c_0 + c_i\theta^i + \frac{1}{2} c_{ij}\theta^i\theta^j + \ldots$$

Then, $\int d\theta_n \cdots d\theta_1 \; P(\theta_1, \ldots, \theta_n)$

$$= \int d\theta_n \cdots d\theta_1 \, \Big(c_0 + \ldots + \underbrace{\frac{1}{n!} c^{(n)}_{ij\ldots} \theta^i\theta^j \cdots}_{= c^{(n)}_{12\ldots n}\theta^1\theta^2\ldots\theta^n}\Big)$$

$$= c^{(n)}_{12\ldots n} \quad \text{(a number.)}$$

Now consider $\tilde{\theta}^i = A^i{}_j \theta^j + \text{const.}$ $\qquad A =$ matrix over Λ^c

Top term $\quad c_{12\ldots n}\theta^1\theta^2\ldots\theta^n \rightarrow \det A \; c_{12\ldots n}\tilde{\theta}^1\tilde{\theta}^2\ldots\tilde{\theta}^n$.

$$\Rightarrow \boxed{d\tilde{\theta}_n \cdots d\tilde{\theta}_1 = (\det A)^{-1} d\theta_1 \cdots d\theta_n} \qquad A^i{}_j = \frac{\partial\tilde{\theta}^i}{\partial\theta^j}$$

Grassmann integration by parts

Let $f(\theta) = f_0 + f_1 \theta$

$g(\theta) = g_0 + g_1 \theta$

Consider

$$\int d\theta \; f(\theta) \frac{\partial}{\partial\theta} g(\theta)$$

$$= \int d\theta \, (f_0 + f_1\theta) \frac{\partial}{\partial\theta} (g_0 + g_1\theta)$$

$$= \int d\theta \, (f_0 + \theta f_1) g_1$$

$$= f_1 g_1$$

compare with

$$\int d\theta \; \frac{\partial}{\partial\theta} f(\theta) \; g(\theta)$$

$$= \int d\theta \; f_1 \, (g_0 + g_1\theta)$$

$$= f_1 g_1$$

equal

$$\Rightarrow \boxed{\int d\theta \, f \frac{\partial}{\partial\theta} g = + \int d\theta \, \frac{\partial f}{\partial\theta} g}$$

Dirac delta

Defined by relation:

$$\int d\theta \; \delta(\theta - \theta') \, f(\theta) = f(\theta')$$

same result obtained by replacement $\boxed{\delta(\theta - \theta') \longrightarrow \theta - \theta'}$

check:

$$\int d\theta \; (\theta - \theta')(f_0 + f_1\theta)$$

$$= \int d\theta \, (\theta f_0 + 0 - \theta' - \theta' f_1 \theta)$$

$$= f_0 + f_1 \theta' \quad \checkmark$$

Derivative of Dirac delta:

$$\boxed{\frac{\partial}{\partial\theta} \delta(\theta - \theta') = -1}$$ (as a distribution in θ)