Lorentz Group

Point in space-time manifold: $x^\mu = (x^0; \vec{x}) \equiv (\underset{\text{time}}{x^0}; \underbrace{x^1, x^2, x^3}_{\text{space}})$

Lorentz transformations $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$ leave quadratic form invariant:

$$x'^2 = x'^\mu x'_\mu = g_{\mu\nu} x'^\mu x'^\nu = g_{\mu\nu}(\Lambda^\mu{}_\rho x^\rho)(\Lambda^\nu{}_\sigma x^\sigma) = g_{\rho\sigma} x^\rho x^\sigma = x^2.$$

metric:

$$g_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}_{\mu\nu}$$

define:

$$\boxed{\begin{gathered} (g_{\mu\nu}\Lambda^\mu{}_\rho)\Lambda^\nu{}_\sigma = g_{\rho\sigma} \\ \text{or} \\ \Lambda^\mu{}_\rho g_{\mu\nu} \Lambda^\nu{}_\sigma = g_{\rho\sigma} \\ \Lambda^T \cdot g \cdot \Lambda = g \end{gathered}}$$

More explicitly,

write $\Lambda = \left(\begin{array}{c|c} \Lambda^0{}_0 & \vec{\Lambda} \\ \hline \vec{\Lambda}' & \overleftrightarrow{\Lambda} \end{array}\right)$ $\quad g = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right)$ $\quad \vec{\Lambda}' \neq \vec{\Lambda}$ for general boost.

So that defining equation is:

$$\Lambda^T g \Lambda = \left(\begin{array}{c|c} \Lambda^0{}_0 & \vec{\Lambda}'^T \\ \hline \vec{\Lambda}^T & \overleftrightarrow{\Lambda}^T \end{array}\right) \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right) \left(\begin{array}{c|c} \Lambda^0{}_0 & \vec{\Lambda} \\ \hline \vec{\Lambda}' & \overleftrightarrow{\Lambda} \end{array}\right)} = g = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\mathbb{1} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \Lambda^0{}_0 & \vec{\Lambda}'^T \\ \hline \vec{\Lambda}^T & \overleftrightarrow{\Lambda}^T \end{array}\right) \left(\begin{array}{c|c} \Lambda^0{}_0 & \vec{\Lambda} \\ \hline -\vec{\Lambda}' & -\overleftrightarrow{\Lambda} \end{array}\right)$$

$$= \left(\begin{array}{c|c} (\Lambda^0{}_0)^2 - \vec{\Lambda}' \cdot \vec{\Lambda}' & \Lambda^0{}_0 \vec{\Lambda} - \vec{\Lambda}'^T \cdot \overleftrightarrow{\Lambda} \\ \hline \vec{\Lambda}^T \Lambda^0{}_0 - \overleftrightarrow{\Lambda}^T \cdot \vec{\Lambda}' & \underbrace{\vec{\Lambda}^T \vec{\Lambda}}_{\text{tensor product}} - \overleftrightarrow{\Lambda}^T \cdot \overleftrightarrow{\Lambda} \end{array}\right) = \left(\begin{array}{c|c} 1 & \\ \hline & -\mathbb{1} \end{array}\right)$$

NOTE: This means $\Lambda = g \Lambda^T g = \left(\begin{array}{c|c} \Lambda^0{}_0 & -\vec{\Lambda}'^T \\ \hline -\vec{\Lambda}^T & \overleftrightarrow{\Lambda}^T \end{array}\right)$

The Lorentz transformations form a Lie group: Lorentz group

$$\begin{matrix}\text{Lorentz}\\ \text{group}\end{matrix} \equiv O(1,3;\mathbb{R}) = \{\Lambda \in GL(4,\mathbb{R}) \mid \Lambda^T g \Lambda = g\}$$

"The Lorentz group, denoted $O(1,3;\mathbb{R})$, is the set of all 4×4 real matrices, satisfying the property $\Lambda^T g \Lambda = g$"
(orthogonal)

Connectedness of Lorentz transformations:

① Take determinant of defining equation $\Lambda^T g \Lambda = g$

$$(\det \Lambda^T)(\det g)(\det \Lambda) = \det g$$

$$(\det \Lambda)\cdot\cancel{(-1)}\cdot(\det \Lambda) = \cancel{-1}$$

$$(\det \Lambda)^2 = +1$$

$$\Rightarrow \boxed{\det \Lambda = +1} \quad \text{or} \quad \boxed{\det \Lambda = -1}$$

② Take $(\mu,\nu) = (0,0)$ component of defining equation $\Lambda^T g \Lambda = g$

$$(\Lambda^0{}_0)^2 - \vec{\Lambda}'\cdot\vec{\Lambda}' = 1$$

$$(\Lambda^0{}_0)^2 = 1 + \underbrace{\vec{\Lambda}'\cdot\vec{\Lambda}'}_{\substack{\text{length-squared}\\ \text{of vector} > 0}}$$

this is just saying time always dilates.

$$\therefore \quad (\Lambda^0{}_0)^2 \geq +1$$

limiting case $\vec{\Lambda}^2 = 0$ occurs for pure rotations (no boost $\Rightarrow$ no time dilation).

$$\Rightarrow \boxed{\Lambda^0{}_0 \geq +1} \quad \text{or} \quad \boxed{\Lambda^0{}_0 \leq -1}$$

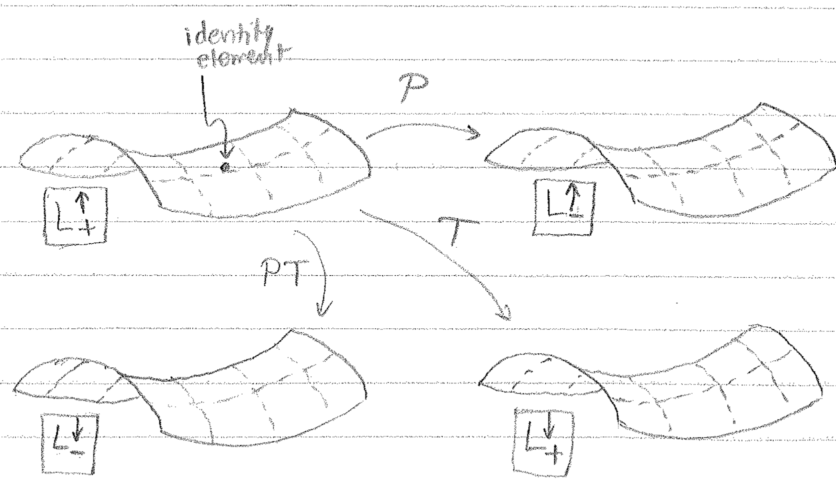
The Lorentz group has four disconnected pieces (subsets),

① labeled by which of the four special elements $\{\mathbb{1}, P, T, PT\}$ they contain.

or ② labeled by the signs of $\Lambda^0{}_0$ and $(\det \Lambda)$.

<u>Classification of the elements of the Lorentz group</u>

	SUBSET	CONTAINS	sgn $\Lambda^0{}_0$	det Λ	
$O(3,1;\mathbb{R})$	Proper orthochronus	$L_+^\uparrow \ni \mathbb{1}$	≥ 1	$+1$	$SO(3,1;\mathbb{R})$
	Improper Non-orthochronus	$L_-^\downarrow \ni PT$	≤ 1	$+1$	
	Improper orthochronus	$L_-^\uparrow \ni P$	≥ 1	-1	
	Proper Non-orthochronus	$L_+^\downarrow \ni T$	≤ 1	-1	



Spatial reflection:

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

time reflection:

$$T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

space-time refl.

$$PT = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

<u>Infinitessimal Lorentz transformations</u>

In the neighborhood of the identity, $\Lambda^\mu{}_\nu$ can be written:

$$\Lambda^\mu{}_\nu = \underbrace{\delta^\mu{}_\nu}_{\text{identity}} + \underbrace{\omega^\mu{}_\nu}_{\text{small transformation.}}$$

Plug into defining equation: $g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$

$$g_{\mu\nu}\left(\delta^\mu{}_\rho + \omega^\mu{}_\rho\right)\left(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma\right) = g_{\rho\sigma}$$

$$\overbrace{\cancel{g_{\rho\sigma}}}^{\text{first}} + \overbrace{g_{\rho\nu}\omega^\nu{}_\sigma}^{\text{outer}} + \overbrace{g_{\mu\sigma}\omega^\mu{}_\rho}^{\text{inner}} + \mathcal{O}(\omega^2) = \cancel{g_{\rho\sigma}}$$

$$\omega_{\rho\sigma} + \omega_{\sigma\rho} = 0$$

$$\boxed{\omega_{\rho\sigma} = -\omega_{\sigma\rho}} \qquad \underline{\text{Antisymmetric.}}$$

Parametrize: $\omega^\mu{}_\nu = -\frac{i}{2}\theta_{\rho\sigma}\left(M^{\rho\sigma}\right)^\mu{}_\nu$

$$\text{so that } \Lambda^\mu{}_\nu = \delta^\mu{}_\nu - \frac{i}{2}\underbrace{\theta_{\rho\sigma}}_{\text{parameters}}\underbrace{\left(M^{\rho\sigma}\right)^\mu{}_\nu}_{\text{generators.}} + \mathcal{O}(\theta^2)$$

Then, all PROPER-ORTHOCHRONUS transformations (connected to identity) can be written as

$$\underline{\Lambda^\mu{}_\nu(\theta) = \left[e^{-\frac{i}{2}\theta_{\rho\sigma}M^{\rho\sigma}}\right]^\mu{}_\nu} \in L_+^\uparrow$$

(active)