$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \eta_1 & \eta_2 & \eta_3 \\ & 0 & \theta_3 & -\theta_2 \\ & & 0 & \theta_1 \\ & & & 0 \end{pmatrix}$$ ← Boosts; } rotations

Representations of the Lorentz Group

Suppose we have a field that carries a generic Lorentz index, $\hat{\phi}_A(x)$.

Group theoretic Transformation: $\hat{\phi}_A(x) \longrightarrow \underbrace{L_A{}^B(\Lambda)}_{\substack{SO(3,1)\text{ Transformation matrix.}\\ \text{(mixes up indices)}}} \hat{\phi}_B(\Lambda^{-1}x)$

Implemented as a Unitary transformation: (in quantum mechanics) $\hat{\phi}_A(x) \longrightarrow \hat{U}(\Lambda)\,\hat{\phi}_A(x)\,\hat{U}^\dagger(\Lambda)$

x acts as an index to label oscillators — not a Q.M. operator $\Rightarrow$ does NOT transform.

The two transformations must match.

$$\hat{U}(\Lambda)\,\hat{\phi}_A(x)\,U^\dagger(\Lambda) = L_A{}^B(\Lambda)\,\hat{\phi}_B(x) \qquad (*)$$

+ sign for (−+++)

Set of matrices describing how components of ϕ_A mix under L.T.

Infinitessimals: $L_A{}^B(1+\delta\theta) = \delta_A{}^B - \frac{i}{2}\theta_{\mu\nu}(S^{\mu\nu})_A{}^B$

$\hat{U}(1+\delta\theta) = \hat{1} + \frac{i}{2}\theta_{\mu\nu}\hat{M}^{\mu\nu}$

$\Lambda^{-1}x = (x^\mu - \theta^\mu{}_\nu x^\nu)$

Operator that serves as the generator for L.T. in quantum mechanics.

Treat parameters of transformation $\delta\omega_{\mu\nu}$ small, and expand $(*)$ to $O(\delta\omega_{\mu\nu})$

$$\left(1+\frac{i}{2}\theta_{\mu\nu}\hat{M}^{\mu\nu}\right)\hat{\phi}_A\left(1-\frac{i}{2}\theta_{\mu\nu}\hat{M}^{\mu\nu}\right) = \left(\delta_A{}^B - \frac{i}{2}\theta_{\mu\nu}(S^{\mu\nu})_A{}^B\right)\hat{\phi}_B(\underbrace{x^\mu - \theta^\mu{}_\nu x^\nu}_{\text{Taylor exp.}})$$

$$\cancel{\hat{\phi}_A} + \frac{i}{2}\theta_{\mu\nu}[\hat{M}^{\mu\nu}, \hat{\phi}_A] = \left(\delta_A{}^B - \frac{i}{2}\theta_{\mu\nu}(S^{\mu\nu})_A{}^B\right)\left[\hat{\phi}_B(x) - \underbrace{\theta^\mu{}_\nu x^\nu \partial_\mu \phi(x)}_{\text{antisymmetrize.}}\right]$$

$$= \cancel{\hat{\phi}_A} \underbrace{- \frac{\theta_{\mu\nu}}{2}(x^\nu\partial^\mu - x^\mu\partial^\nu)\hat{\phi}_A}_{+\frac{\theta_{\mu\nu}}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu)\hat{\phi}_A} - \frac{i}{2}\theta_{\mu\nu}(S^{\mu\nu})_A{}^B\hat{\phi}_B$$

Divide by $\frac{i}{2}$:

$$\theta_{\mu\nu}\left\{[\hat{M}^{\mu\nu}, \hat{\phi}_A] = \underbrace{-i(x^\mu\partial^\nu - x^\nu\partial^\mu)\hat{\phi}_A}_{\substack{\text{always present for}\\ \text{all fields.}}} - (S^{\mu\nu})_A{}^B\hat{\phi}_B\right\}$$

Recall, $\delta\hat{\phi}_A = i\epsilon[\hat{G}, \hat{\phi}_A]$

$\epsilon = \frac{\theta_{\mu\nu}}{2}$ in our case.