Transformation law of spinors χ_a χ^a $\chi^{\dagger\dot a}$ $\chi^\dagger_{\dot a}$ (not spinor fields, so no x^μ label.)

Start from $\boxed{\chi_a \longrightarrow L(\Lambda)_a{}^b\,\chi_b = \left[e^{-\frac{i}{2}\theta_{\mu\nu}(S_L^{\mu\nu})}\right]_a{}^b \chi_b\,,}$

raise index by multiplying at left by ϵ^{da}.

- write exp as Taylor series: $\exp \sim \delta + \frac{-i}{2}\theta(S_L) + \frac{1}{2}\left(\frac{-i}{2}\right)^2\theta^2(S_L)(S_L) + \dots$
- use following identity:

$$\epsilon^{ab}(S_L^{\mu\nu})_b{}^d = \epsilon^{ab}\frac{i}{2}\sigma^{(\mu}_{b\dot c}\bar\sigma^{\nu)\dot c d}$$
$$= \frac{i}{2}\bar\sigma^{(\mu|\dot e a}\epsilon_{\dot c\dot e}\bar\sigma^{\nu)\dot c d}$$
$$= \frac{-i}{2}\bar\sigma^{(\mu|\dot e a}\sigma^{\nu)}_{f\dot e}\epsilon^{df}$$
$$= \frac{i}{2}\sigma^{(\nu}_{f\dot e}\bar\sigma^{\mu)\dot e a}\epsilon^{df}$$
$$= (S_L^{\nu\mu})_b{}^a\epsilon^{bd}$$

$$\Rightarrow \underline{\underline{\epsilon^{ab}(S_L^{\mu\nu})_b{}^d = -(S_L^{\mu\nu})_b{}^a\epsilon^{bd}}}$$ ⟵ use this to push ϵ through S_L (at a cost of minus sign)

So then,

$$\boxed{\chi^d \longrightarrow \left[\delta_a{}^d + \frac{i}{2}\theta_{\mu\nu}(S_L^{\mu\nu})_a{}^d + \dots\right]\underbrace{\epsilon^{ab}\chi_b}_{\chi^a} = \chi^a\underbrace{\left[e^{+\frac{i}{2}\theta_{\mu\nu}(S_L^{\mu\nu})}\right]_a{}^d}_{=R^\dagger(\Lambda)_a{}^d}}$$

to front

To get transformation law for $\chi^{\dagger\dot a}$, take dagger of this

Notice:

$$(\sigma^\mu\bar\sigma^\nu)_a{}^{c\,\dagger} = (\sigma^\mu_{a\dot b}\bar\sigma^{\nu\dot b c})^\dagger = (\bar\sigma^{\nu\dot b c})(\sigma^\mu_{a\dot b})^\dagger = \bar\sigma^{\nu\dot c b}\sigma^\mu_{b\dot a} = (\bar\sigma^\nu\sigma^\mu)^{\dot c}{}_{\dot a}$$

Therefore, $(S_L^{\mu\nu})^\dagger = \left[\frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)\right]^\dagger = \frac{-i}{4}(\bar\sigma^\nu\sigma^\mu - \bar\sigma^\mu\sigma^\nu)$

$= \frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu) = S_R^{\mu\nu}$ $\qquad \underline{(S_L^{\mu\nu})^\dagger = S_R^{\mu\nu}}$

$$\Rightarrow \boxed{\chi^{d\dagger} \longrightarrow \left[e^{-\frac{i}{2}\theta_{\mu\nu}(S_R^{\mu\nu})}\right]^{\dot d}{}_{\dot a}\chi^{\dagger\dot a} = R(\Lambda)^{\dot d}{}_{\dot a}\chi^{\dagger\dot a}}$$

and similarly,

$$\boxed{\chi^\dagger_{\dot d} \longrightarrow \chi^\dagger_{\dot a}\left[e^{+\frac{i}{2}\theta_{\mu\nu}(S_R^{\mu\nu})}\right]^{\dot a}{}_{\dot d} = \chi^\dagger_{\dot a}L^\dagger(\Lambda)^{\dot a}{}_{\dot d}}$$

$\Rightarrow$ Natural $(\frac{1}{2}, 0)$ spinor is χ_a (index down)
'' $(0, \frac{1}{2})$ spinor is $\chi^{\dagger \dot{a}}$ (index up)
because generators act from the left: $J_z \chi = \begin{pmatrix} +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

If $\chi_a = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$, then $\chi^a = \epsilon^{ab}\chi_b = \begin{pmatrix} \chi_2 \\ -\chi_1 \end{pmatrix} \equiv \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix}$

similarly, $\chi^{\dagger \dot{a}} = \begin{pmatrix} \chi^{\dagger \dot{1}} \\ \chi^{\dagger \dot{2}} \end{pmatrix}$, then $\chi^\dagger_{\dot{a}} = \epsilon_{\dot{a}\dot{b}} \chi^{\dagger \dot{b}} = \begin{pmatrix} -\chi^{\dagger \dot{2}} \\ \chi^{\dagger \dot{1}} \end{pmatrix} \equiv \begin{pmatrix} \chi^\dagger_{\dot{1}} \\ \chi^\dagger_{\dot{2}} \end{pmatrix}$

Explicit forms of simple contractions

$$\chi^a \xi_a = \epsilon^{ab} \chi_b \xi_a = \chi_2 \xi_1 - \chi_1 \xi_2$$

[If χ & ξ commute, $\chi\chi = \xi\xi = 0$]

$$= \chi^a \epsilon_{ab} \xi^b = -\chi^1 \xi^2 + \chi^2 \xi^1$$

Two component spinors and helicity state interactions

Interactions of the form $-g\, \chi^\dagger A_\mu \bar{\sigma}^\mu \chi$.

— Define polarization vectors: $\epsilon_L = \frac{1}{\sqrt{2}}(\epsilon^0 + \epsilon^3)$, $\epsilon_+ = \frac{1}{\sqrt{2}}(\epsilon^1 + i\epsilon^2)$, $\epsilon_T = \frac{1}{\sqrt{2}}(\epsilon^0 - \epsilon^3)$, $\epsilon_- = \frac{1}{\sqrt{2}}(\epsilon^1 - i\epsilon^2)$ — gauge boson moving along $\hat{3}$ dir.

— Define $\chi_a = \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix}$ helicity states up and down (e-states of J_z) — fermion moving along $\hat{3}$ dir.

Then, using $\epsilon_\mu \bar{\sigma}^\mu = \epsilon^0 \bar{\sigma}^0 - \vec{\epsilon} \cdot \vec{\sigma} = \sqrt{2} \begin{pmatrix} \epsilon_L & \epsilon_+ \\ \epsilon_- & \epsilon_T \end{pmatrix}$,
we have:

$$\chi^\dagger_{\dot{a}} A_\mu \bar{\sigma}^{\mu \dot{a} b} \chi_b \sim (\chi^\dagger_\uparrow \;\; \chi^\dagger_\downarrow) \sqrt{2} \begin{pmatrix} \epsilon_L & \epsilon_+ \\ \epsilon_- & \epsilon_T \end{pmatrix} \begin{pmatrix} \chi_\uparrow \\ \chi_\downarrow \end{pmatrix}$$

$$= \sqrt{2} \left(\chi^\dagger_\uparrow \epsilon_L \chi_\uparrow + \chi^\dagger_\uparrow \epsilon_+ \chi_\downarrow + \chi^\dagger_\downarrow \epsilon_- \chi_\uparrow + \chi^\dagger_\downarrow \epsilon_T \chi_\downarrow \right)$$