Spin

— see Leader's (2001) "Spin in Particle Physics" for more details.

In a manner similar to ordinary QM,

$$\underset{\substack{\uparrow \\ \text{axis of} \\ \text{quantization}}}{\vec{s}}\cdot\hat{\vec{J}}\,|s, m_s\rangle = m_s\,|s, m_s\rangle$$

Can go back here by going to rest frame.

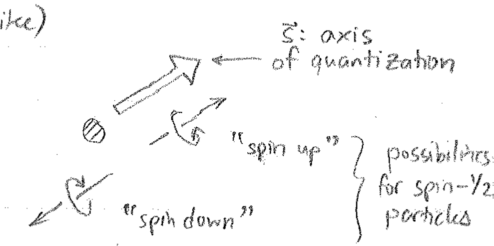
can generalize to relativistic systems:

$$\underset{\substack{\uparrow \\ \text{due to metric} \\ (+,-,-,-)}}{-}\, S_\mu W^\mu |s, m_s\rangle = m_s |s, m_s\rangle$$

W^μ: Pauli-Lubanski vector

$S_\mu \equiv$ axis* of quantization (axial vector) $\longleftarrow \underbrace{(0; \vec{s})}_{\text{Your choice (axis)}}$ in rest frame

Must have $\underline{S_\mu p^\mu = 0}$, and $S_\mu S^\mu = -1$ (space-like)



Possible Axes of Quantization, S_μ

① As in elementary quantum mechanics,

Along z-axis: $\vec{s} = (0, 0, 1)$ in rest frame.

— can choose different direction simply by rotating $\vec{s}$:

Fixed-axis spin

$$\vec{s} \longrightarrow \hat{s} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

Can build 4-vector S^μ appropriate for moving particles with momentum $\vec{p}$ by actively boosting in positive $\vec{p}$-direction.

$$\underset{\substack{\uparrow \\ \text{Moving} \\ \text{frame}}}{S^\mu} = \underbrace{\Lambda^\mu{}_\nu(p)}_{\substack{\uparrow \\ \text{active} \\ \text{boost}}} \underset{\substack{\uparrow \\ \text{Rest} \\ \text{frame}}}{S^\nu} = \underbrace{\begin{pmatrix} \frac{\omega_p}{m} & \frac{p_j}{m} \\ \frac{p_i}{m} & \delta_{ij} + \frac{p_i p_j}{m(\omega_p + m)} \end{pmatrix}}_{\text{active boost in } +\vec{p} \text{ direction}} \begin{pmatrix} 0 \\ \hline s_j \end{pmatrix} = \begin{pmatrix} \frac{1}{m}\vec{p}\cdot\hat{s} \\ \hat{s} + \frac{(\vec{p}\cdot\hat{s})\,\vec{p}}{m(\omega_p + m)} \end{pmatrix}$$

$\hat{s} \equiv$ unit vector defined in rest frame.

---

*Not to be confused with 3-vector $\vec{s}$ along which system is pointing up (independent of axis of quantization)

② Helicity (quantization axis along 3-momentum)

$$\hat{s} = \frac{\vec{p}}{|\vec{p}|} \equiv \hat{p}$$ (In the limit the system w/ momentum $\vec{p}$ is brought to rest)

In this case

$$S^\mu = \begin{pmatrix} \frac{1}{m}\vec{p}\cdot\hat{p} \\ \hat{p} + \frac{(\vec{p}\cdot\hat{p})\vec{p}}{m(\omega_p+m)} \end{pmatrix} = \begin{pmatrix} \frac{1}{m}|\vec{p}| \\ \hat{p} + \frac{|\vec{p}|^2\,\hat{p}}{m(\omega_p+m)} \end{pmatrix} = \frac{1}{m}\begin{pmatrix} |\vec{p}| \\ \omega_p\,\hat{p} \end{pmatrix}$$

For massless particles, S^μ does not exist (as there is no rest frame in which to define $\hat{s}$). Instead use helicity with

$$\underset{\substack{\uparrow \\ \text{must be space-like}}}{S^\mu} \approx \frac{1}{m}\underset{\substack{\uparrow \\ \text{time-like}}}{p^\mu} + \underset{\substack{\uparrow \\ \text{correction to push off light-cone.}}}{\mathcal{O}\left(\frac{m}{E}\right)}$$

$m \equiv$ not mass, but a small dimensionful parameter.

limit $m \to 0$ should be well behaved at end of calculation.

So, the generalized spin operator (for relativistic systems) is:

$$-S_\mu W^\mu = +\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} S_\mu M_{\nu\rho} P_\sigma$$

$$= \begin{pmatrix} \text{very complicated} \\ \text{for arbitrary } \hat{s} \end{pmatrix} \longrightarrow m(\vec{s}\cdot\vec{J}) \text{ in limit } p^\mu \to (m;\vec{0})$$

For $\hat{s} = \hat{p}$ (helicity), we have, in the rest frame limit:

$$-S_\mu W^\mu = m(\hat{\vec{p}}\cdot\vec{J})$$ (see print-out)

So, quantize system (chose rest-frame basis) using this operator.

Generalized Spin Operator

Using the Pauli-Lubanski 4-vector

$M_{\nu\rho}$

$$\mathtt{M} = \begin{pmatrix} 0 & \mathtt{K1} & \mathtt{K2} & \mathtt{K3} \\ -\mathtt{K1} & 0 & \mathtt{J3} & -\mathtt{J2} \\ -\mathtt{K2} & -\mathtt{J3} & 0 & \mathtt{J1} \\ -\mathtt{K3} & \mathtt{J2} & -\mathtt{J1} & 0 \end{pmatrix};$$

P_σ

```
P = {ωp, p1, p2, p3};
```

① $\vec{s} = (0,0,1)$

$$\mathtt{genS} = \left\{\mathtt{p1*s1+p2*s2+p3*s3},\ \mathtt{s1} + \frac{(\mathtt{p1*s1+p2*s2+p3*s3})\ \mathtt{p1}}{\mathtt{m}\ (\omega\mathtt{p}+\mathtt{m})},\ \mathtt{s2} + \frac{(\mathtt{p1*s1+p2*s2+p3*s3})\ \mathtt{p2}}{\mathtt{m}\ (\omega\mathtt{p}+\mathtt{m})},\ \mathtt{s3} + \frac{(\mathtt{p1*s1+p2*s2+p3*s3})\ \mathtt{p3}}{\mathtt{m}\ (\omega\mathtt{p}+\mathtt{m})}\right\};$$

② $\vec{s} = \hat{p}$

$$\mathtt{helS} = \frac{1}{\mathtt{m}}\left\{\sqrt{\mathtt{p1}^2+\mathtt{p2}^2+\mathtt{p3}^3},\ \omega\mathtt{p}\frac{\mathtt{p1}}{\sqrt{\mathtt{p1}^2+\mathtt{p2}^2+\mathtt{p3}^3}},\ \omega\mathtt{p}\frac{\mathtt{p2}}{\sqrt{\mathtt{p1}^2+\mathtt{p2}^2+\mathtt{p3}^3}},\ \omega\mathtt{p}\frac{\mathtt{p3}}{\sqrt{\mathtt{p1}^2+\mathtt{p2}^2+\mathtt{p3}^3}}\right\};$$

① $S_\mu W^\mu$ in rest frame limit

$$\mathtt{Sum}\left[-\frac{1}{2\,\mathtt{m}}\ \mathtt{Signature[\{i, j, k, l\}]*genS[[i]]*M[[j, k]]*P[[l]]},\ \mathtt{\{i, 1, 4\}, \{j, 1, 4\}, \{k, 1, 4\}, \{l, 1, 4\}}\right]\ \mathtt{/.}\ \{\omega\mathtt{p}\to\mathtt{m},\ \mathtt{p1}\to 0,\ \mathtt{p2}\to 0,\ \mathtt{p3}\to 0\}$$

$$\mathrm{J1\ s1 + J2\ s2 + J3\ s3}$$

② $S_\mu W^\mu$ (helicity) in rest frame limit.

$$\mathtt{Simplify}\left[\mathtt{Sum}\left[-\frac{1}{2\,\mathtt{m}}\ \mathtt{Signature[\{i, j, k, l\}]*helS[[i]]*M[[j, k]]*P[[l]]},\ \mathtt{\{i, 1, 4\}, \{j, 1, 4\}, \{k, 1, 4\}, \{l, 1, 4\}}\right]\right]\ \mathtt{/.}\ \mathtt{p1}^2+\mathtt{p2}^2+\mathtt{p3}^3-\omega\mathtt{p}^2\to-\mathtt{m}^2$$

$$\frac{\mathrm{J1\ p1 + J2\ p2 + J3\ p3}}{\sqrt{\mathrm{p1}^2+\mathrm{p2}^2+\mathrm{p3}^3}} = \frac{\vec{p}\cdot\vec{J}}{|\vec{p}|} = \hat{p}\cdot\vec{J}$$