Quantization of a left-handed Weyl Field

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi - \frac{1}{2}m(\chi\chi + \chi^\dagger\chi^\dagger)$$

$$= i\chi^\dagger_{\dot{a}} \bar{\sigma}^{\mu \dot{a} b} \partial_\mu \chi_b - \frac{1}{2}m(\chi^a\chi_a + \chi^\dagger_{\dot{a}}\chi^{\dagger\dot{a}})$$

Extremize action to obtain E-L equation of motion.

$$S = \int d^4x \, \mathcal{L}[\chi, \partial_\mu\chi; \chi^\dagger, \partial_\mu\chi^\dagger]$$

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \chi}\delta\chi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\chi)}\delta(\partial_\mu\chi) + \delta\chi^\dagger \frac{\partial\mathcal{L}}{\partial\chi^\dagger} + \delta(\partial_\mu\chi^\dagger)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi^\dagger)} \right) = 0$$

integrate by parts in 2^{nd} and 4^{th} terms.

$$= \int d^4x \left[\left(\frac{\partial\mathcal{L}}{\partial\chi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_a)} \right)\delta\chi_a + \delta\chi^\dagger_{\dot{a}} \left(\frac{\partial\mathcal{L}}{\partial\chi^\dagger_{\dot{a}}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi^\dagger_{\dot{a}})} \right) \right] = 0$$

Hence, the two equations of motion are:

$$\boxed{\frac{\partial\mathcal{L}}{\partial\chi_a} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi_a)} = 0} \quad \text{AND} \quad \boxed{\frac{\partial\mathcal{L}}{\partial\chi^\dagger_{\dot{a}}} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi^\dagger_{\dot{a}})} = 0}$$

By definition, the canonical momenta are:

$$\pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)} = i\chi^\dagger\bar{\sigma}^\mu \qquad \text{(derivative acts from right)}$$

$$\pi^{\dagger\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi^\dagger)} = 0 \qquad \leftarrow \text{Lagrangian not symmetrized.}$$

To Hamiltonian

does not depend on velocities. $\Rightarrow$ CONSTRAINTS.

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\chi}} = i\chi^\dagger \bar{\sigma}^0 \longrightarrow \phi_1^{\,a} = (\pi - i\chi^\dagger \bar{\sigma}^0)^a = 0$$

$$\pi^\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\chi}^\dagger} = 0 \longrightarrow \phi_2^{\dagger\dot{a}} = \pi^{\dagger\dot{a}} = 0$$

Note: these constraints are Grassman valued, and carry a $(1/2, 0)$ Lorentz index.

Canonical Hamiltonian: "Naive Hamiltonian"

$$H_{can} = \int d^3x \left[\dot{\chi}^\dagger \pi^\dagger + \pi \dot{\chi} - \mathcal{L} \right]$$

$$= \int d^3x \left[0 + i\chi^\dagger \bar{\sigma}^0 \partial_0 \chi - i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi + \frac{1}{2} m(\chi\chi + \chi^\dagger\chi^\dagger) \right]$$

$\mu = 0$ term cancels.

$$= \int d^3x \left[- i\chi^\dagger \bar{\sigma}_i \nabla_i \chi + \frac{1}{2} m(\chi\chi + \chi^\dagger\chi^\dagger) \right] \qquad (*)$$

no extra minus sign from metric because $\partial_\mu = \left(\frac{\partial}{\partial t}, + \vec{\nabla}\right)$

$$= \int d^3x \left[+ i\chi^\dagger \vec{\bar{\sigma}} \cdot \vec{\nabla} \chi + \frac{1}{2} m(\chi\chi + \chi^\dagger\chi^\dagger) \right].$$

Will use form in $(*)$ to carry out analysis.
Then, the extended Hamiltonian is:

$$H_{ext} = H_{can} + \int d^3x \left[\phi_1 u_1 + u_2^\dagger \phi_2^\dagger \right]$$

$$= \int d^3x \left[-i\chi^\dagger \bar{\sigma}_i \nabla_i \chi + \frac{1}{2} m(\chi\chi + \chi^\dagger\chi^\dagger) + (\pi - i\chi^\dagger \bar{\sigma}^0) u_1 + u_2^\dagger \pi^\dagger \right]$$

Require time independence of constraints (on-shell)

$$\dot{\phi}_1^a(x) = \{\phi_1(x), H_{ext}\} \approx 0$$

for Fermions

$$= \int d^3z \left(\frac{\delta \phi_1^a(x)}{\delta \chi_b(z)} \frac{\delta H_{ext}}{\delta \pi^b(z)} + \frac{\delta H_{ext}}{\delta \pi^{\dagger b}} \frac{\delta \phi_1}{\delta \chi^\dagger_b} + \frac{\delta \phi_1^a}{\delta \pi_b} \frac{\delta H_{ext}}{\delta \chi^b} + \frac{\delta H_{ext}}{\delta \chi^{\dagger b}} \frac{\delta \phi_1}{\delta \pi^\dagger_b} \right)$$

switched for convenience.

use $\phi_1(x) = \pi - i\chi^\dagger \bar{\sigma}^0$

$$= \int d^3z \left(-u_2^\dagger (-i\bar{\sigma}^0 \delta^{(3)}(x-z)) + \left[(-\nabla_i)(-i\chi^\dagger \bar{\sigma}_i) + m\chi \right] \delta^{(3)}(x-z) \right)$$

$$= i u_2^\dagger \bar{\sigma}^0 + i \nabla_i \chi^\dagger \bar{\sigma}_i - m\chi \approx 0$$

This equation serves to fix $u_2^\dagger$:

$$\underline{u_2 = -\nabla_i \chi^\dagger \bar{\sigma}_i \sigma^0 - i m \chi \sigma^0}$$

$$\dot{\phi}_2^a(x) = \{\phi_2(x), H_{ext}\} \approx 0$$

$$= \int d^3z \left(\frac{\delta \phi_2^a(x)}{\delta \chi_b(z)} \frac{\delta H_{ext}}{\delta \pi^b(z)} + \frac{\delta H_{ext}}{\delta \pi^{\dagger b}} \frac{\delta \phi_2}{\delta \chi^\dagger_b} + \frac{\delta \phi_2^a}{\delta \pi_b} \frac{\delta H_{ext}}{\delta \chi^b} + \frac{\delta H_{ext}}{\delta \chi^{\dagger b}} \frac{\delta \phi_1}{\delta \pi^\dagger_b} \right)$$

$$= \int d^3z \left(\delta^{(3)}(x-z) \left(-i\bar{\sigma}_i \nabla_i \chi + m\chi^\dagger - i\bar{\sigma}^0 u_1 \right) \right)$$

$$= -i\bar{\sigma}_i \nabla_i \chi + m\chi^\dagger - i\bar{\sigma}^0 u_1 \approx 0$$

This equation serves to fix u_1

$$\underline{u_1 = \sigma^0 \bar{\sigma}_i \nabla_i \chi + i m \sigma^0 \chi^\dagger}$$

Thus, there are no more constraints

The two constraints are second class:

$$\{\phi_1(x), \delta\phi_2^\dagger(y)\} = \int d^3z \left(\frac{\delta\phi_1}{\delta\chi}\frac{\delta\phi_2^\dagger}{\delta\pi} + \frac{\delta\phi_1}{\delta\chi^\dagger}\frac{\delta\phi_2^\dagger}{\delta\pi^\dagger} + \frac{\delta\phi_1}{\delta\pi}\frac{\delta\phi_2^\dagger}{\delta\chi} + \frac{\delta\phi_1}{\delta\pi^\dagger}\frac{\delta\phi_2^\dagger}{\delta\chi^\dagger}\right)$$

$$= \int d^3z \left(-i\bar{\sigma}^0\, \delta(\vec{x}-\vec{z})\,\delta(\vec{y}-\vec{z})\right)$$

$$= -i\bar{\sigma}^0\,\delta(\vec{x}-\vec{y}) \quad \checkmark$$

Matrix of Poisson brackets is: Inverse:

$$\{\phi_a, \phi_b\} = \begin{pmatrix} 0 & -i\bar{\sigma}^0 \\ -i\bar{\sigma}^0 & 0 \end{pmatrix} \delta^{(3)}(\vec{x}-\vec{y}) \qquad \{\phi_a, \phi_b\}^{-1} = \begin{pmatrix} 0 & i\bar{\sigma}^0 \\ i\bar{\sigma}^0 & 0 \end{pmatrix}\delta^{(3)}(\vec{x}-\vec{y})$$

Then, the <u>Dirac bracket</u> between two dynamical quantities f & g is:

$$\{f, g\}_{D.B.} = \{f, g\} - \sum_{a,b} \{f, \phi_a\}\{\phi_a, \phi_b\}^{-1}\{\phi_b, g\}$$

$$\{f, \phi_1(x)\} = \int d^3z \left(\frac{\delta f}{\delta\chi}\delta^{(3)}(x-z) + \frac{\delta f}{\delta\pi^\dagger} i\bar{\sigma}^0\,\delta^{(3)}(x-z)\right)$$

$$= \frac{\partial f}{\partial\chi} + i\frac{\delta f}{\delta\chi^\dagger}\bar{\sigma}^0 \qquad \{g, \phi_1\} = \frac{\partial g}{\partial\chi} + i\frac{\partial g}{\partial\chi}\bar{\sigma}^0$$

$$\{f, \phi_2(x)\} = \int d^3z \left(\frac{\delta f}{\delta\chi^\dagger}\delta^{(3)}(x-z)\right)$$

$$= \frac{\partial f}{\partial\chi^\dagger} \qquad \{g, \phi_2\} = \frac{\partial g}{\partial\chi^\dagger}$$

$$\{f, g\}_{D.B} = \{f, g\} - \begin{pmatrix}\frac{\partial f}{\partial\chi} + i\frac{\partial f}{\partial\chi^\dagger}\bar{\sigma}^0 & \frac{\partial f}{\partial\chi^\dagger}\end{pmatrix}\begin{pmatrix} 0 & i\bar{\sigma}^0 \\ i\bar{\sigma}^0 & 0\end{pmatrix}\begin{pmatrix}\frac{\partial g}{\partial\chi} + i\frac{\partial g}{\partial\chi}\bar{\sigma}^0 \\ \frac{\partial g}{\partial\chi^\dagger}\end{pmatrix}$$

$$= \{f, g\} - \left(\frac{\partial f}{\partial\chi} + i\frac{\partial f}{\partial\chi^\dagger}\bar{\sigma}^0\right) i\bar{\sigma}^0\frac{\partial g}{\partial\chi^\dagger} - \frac{\partial f}{\partial\chi^\dagger}\left(i\bar{\sigma}^0\frac{\partial g}{\partial\chi} + i\frac{\partial g}{\partial\chi}\bar{\sigma}^0\right)$$

So, the Dirac bracket of the two canonical variables χ and $\pi = i\chi^\dagger \bar{\sigma}^0$ is.

$$\{\chi, i\chi^\dagger \bar{\sigma}^0\}_{DB} = 0 - \left(\delta^{(3)}(x-y) + 0\right) i\bar{\sigma}^0 \, i\bar{\sigma}^0 - 0$$

$$= \delta^{(3)}(x-y)$$

To pass to quantum mechanics, upgrade Dirac Brackets to (anti) commutators.
(and multiply by $i\hbar$)

$$\{\chi, i\chi^\dagger \bar{\sigma}^0\} = i\hbar \, \delta^{(3)}(x-y)$$

then set the constraints to zero in Hamiltonian:

$$\hat{H} = \int d^3x \left[-i\hat{\chi}^\dagger \bar{\sigma}_i \nabla_i \hat{\chi} + \frac{1}{2} m \left(\hat{\chi}\hat{\chi} + \hat{\chi}^\dagger \hat{\chi}^\dagger \right) \right]$$