Generators of Lorentz group

Group: $SO(3,1)$ — metric $\begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

Group of 4×4 matices, $\Lambda^{\mu}{}_{\nu}$

Special: $\det(\Lambda) = 1$

Orothogonal: $\Lambda\Lambda^T = 1$

(Active)
Transformation law of 4-vector: $v^{\mu} \longrightarrow \Lambda(\theta)^{\mu}{}_{\nu}\, v^{\nu} = \left(e^{-\frac{i}{2}\theta_{\rho\sigma} M^{\rho\sigma}}\right)^{\mu}{}_{\nu}\, v^{\nu}$

SIX Parameters — $\theta_{\rho\sigma}$ (antisymmetric $\theta_{\rho\sigma} = -\theta_{\sigma\rho}$)

$$\left.\begin{aligned} \theta_{12} &= \theta_z \\ \theta_{13} &= -\theta_y \\ \theta_{23} &= \theta_x \end{aligned}\right\} \text{ROTATION} \qquad \left.\begin{aligned} \theta_{01} &= \eta_x \\ \theta_{02} &= \eta_y \\ \theta_{03} &= \eta_z \end{aligned}\right\} \text{RAPIDITY} \left(\begin{array}{c}\text{parameters} \\ \text{of L. boosts}\end{array}\right)$$

SIX Generators — $M^{\rho\sigma}$ (also antisymmetric)

– satisfy Lorentz algebra:

$$[\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = -i\left(g^{\mu\rho}\hat{M}^{\nu\sigma} - g^{\mu\sigma}\hat{M}^{\nu\rho} - g^{\nu\rho}\hat{M}^{\mu\sigma} + g^{\nu\sigma}\hat{M}^{\mu\rho}\right)$$ – derived later.

Arranged: $M^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & K_x & K_y & K_z \\ \hline & 0 & J_z & -J_y \\ & & 0 & J_x \\ & & & 0 \end{array}\right)$ c.f. $F^{\mu\nu} = \left(\begin{array}{c|ccc} & -E_x & -E_y & -E_z \\ \hline & & -B_z & B_y \\ & & & -B_x \\ & & & \end{array}\right)$ of E/M.

Defining (vector) rep: $M^{\rho\sigma} = (S_V^{\rho\sigma})^{\mu}{}_{\nu} = -i\left(\delta^{\rho}{}_{\nu}\, g^{\sigma\mu} - \delta^{\sigma}{}_{\nu}\, g^{\rho\mu}\right)$

$\rho, \sigma \leftarrow$ label generators
$\mu, \nu \leftarrow$ label components

explicitly,

Rotations
$J_i = \frac{1}{2}\epsilon^{ijk} M^{jk}$

$$J_x = \left(\begin{array}{c|ccc} 0 & & & \\ \hline & 0 & & \\ & & 0 & -i \\ & & i & 0 \end{array}\right) \quad J_y = \left(\begin{array}{c|ccc} 0 & & & \\ \hline & 0 & & i \\ & & 0 & \\ & -i & & 0 \end{array}\right) \quad J_z = \left(\begin{array}{c|ccc} 0 & & & \\ \hline & 0 & -i & \\ & i & 0 & \\ & & & 0 \end{array}\right)$$

Boosts
$K_i = M^{0i}$

$$K_x = \left(\begin{array}{c|ccc} 0 & i & 0 & 0 \\ \hline i & & & \\ 0 & & & \\ 0 & & & \end{array}\right) \quad K_y = \left(\begin{array}{c|ccc} 0 & 0 & i & 0 \\ \hline 0 & & & \\ i & & & \\ 0 & & & \end{array}\right) \quad K_z = \left(\begin{array}{c|ccc} 0 & 0 & 0 & i \\ \hline 0 & & & \\ 0 & & & \\ i & & & \end{array}\right)$$ (not antisymm.)

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad [J_i, K_j] = i\epsilon_{ijk} K_k \qquad [K_i, K_j] = -i\epsilon_{ijk} J_k$$

```
metricG[μ_, ν_] = If[μ == ν && μ == 0, 1, If[μ == ν, -1, 0]];
Table[metricG[μ, ν], {μ, 0, 3}, {ν, 0, 3}] // MatrixForm
```

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

```
Clear[σ]
S = Table[Table[-ⅈ (KroneckerDelta[ρ, ν] metricG[σ, μ] - KroneckerDelta[σ, ν] metricG[ρ, μ]),
    {μ, 0, 3}, {ν, 0, 3}], {ρ, 0, 3}, {σ, 0, 3}];
Table[S[[ρ, σ]] // MatrixForm, {ρ, 1, 4}, {σ, 1, 4}] // MatrixForm
```

$$M^{\mu\nu} = \begin{pmatrix}
\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&i&0\\0&0&0&0\\i&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&i\\0&0&0&0\\0&0&0&0\\i&0&0&0 \end{pmatrix} \\
\begin{pmatrix} 0&-i&0&0\\-i&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&-i\\0&0&0&0\\0&i&0&0 \end{pmatrix} \\
\begin{pmatrix} 0&0&-i&0\\0&0&0&0\\-i&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&i&0\\0&-i&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0 \end{pmatrix} \\
\begin{pmatrix} 0&0&0&-i\\0&0&0&0\\0&0&0&0\\-i&0&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&i\\0&0&0&0\\0&-i&0&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&i\\0&0&-i&0 \end{pmatrix} & \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}
\end{pmatrix}$$

← Boosts

Rotations

$$Jx = \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0 \end{pmatrix};\ Jy = \begin{pmatrix} 0&0&0&0\\0&0&0&i\\0&0&0&0\\0&-i&0&0 \end{pmatrix};\ Jz = \begin{pmatrix} 0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0 \end{pmatrix};$$

$$Kx = \begin{pmatrix} 0&i&0&0\\i&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix};\ Ky = \begin{pmatrix} 0&0&i&0\\0&0&0&0\\i&0&0&0\\0&0&0&0 \end{pmatrix};\ Kz = \begin{pmatrix} 0&0&0&i\\0&0&0&0\\0&0&0&0\\i&0&0&0 \end{pmatrix};$$

Checking commutation relations

```
{Jx.Jy - Jy.Jx == ⅈ Jz,
 Jy.Jz - Jz.Jy == ⅈ Jx,
 Jz.Jx - Jx.Jz == ⅈ Jy}
```

$[J_i, J_j] = i\epsilon_{ijk} J_k$

```
{True, True, True}
```

```
{Jx.Ky - Ky.Jx == ⅈ Kz,
 Jy.Kz - Kz.Jy == ⅈ Kx,
 Jz.Kx - Kx.Jz == ⅈ Ky}
```

$[J_i, K_j] = i\epsilon_{ijk} K_k$

```
{True, True, True}
```

```
{Kx.Ky - Ky.Kx == -ⅈ Jz,
 Ky.Kz - Kz.Ky == -ⅈ Jx,
 Kz.Kx - Kx.Kz == -ⅈ Jy}
```

$[K_i, K_j] = -i\epsilon_{ijk} J_k$

```
{True, True, True}
```