Noether's theorem for Weyl Fermions

Lagrangian: $\mathcal{L} = \mathcal{L}[\chi, \partial_\mu \chi, \chi^\dagger, \partial_\mu \chi^\dagger]$ present in symmetrized Lagrangian.

Transformation law of χ: $\chi \to \chi + \delta\chi$, $\chi^\dagger \to \chi^\dagger + \delta\chi^\dagger$

Then, the Lagrangian transforms: $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$

where

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \chi_\alpha}\delta\chi_\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi_\alpha)}\delta(\partial_\mu \chi_\alpha) + \delta\chi^\dagger_{\dot\alpha}\frac{\partial \mathcal{L}}{\partial \chi^\dagger_{\dot\alpha}} + \delta(\partial_\mu \chi^\dagger_{\dot\alpha})\frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi^\dagger_{\dot\alpha})} \qquad (*)$$

Equations of Motion: $\frac{\partial \mathcal{L}}{\partial \chi_\alpha} = \partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi_\alpha)}}_{\text{abbr: } \pi^{\alpha\mu}}$ and $\frac{\partial \mathcal{L}}{\partial \chi^\dagger_{\dot\alpha}} = \partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi^\dagger_{\dot\alpha})}}_{\text{abbr: } \pi^{\dagger\dot\alpha\mu}}$ = (vanishes for unsymmetrized Lagr.)

then,

$$\delta\mathcal{L} = (\partial_\mu \pi^{\alpha\mu})\delta\chi_\alpha + \pi^{\alpha\mu}(\partial_\mu \delta\chi_\alpha) + \delta\chi^\dagger_{\dot\alpha}(\partial_\mu \pi^{\dagger\dot\alpha\mu}) + (\partial_\mu \delta\chi^\dagger_{\dot\alpha})\pi^{\dagger\dot\alpha\mu}$$

$$= \partial_\mu\Big[\pi^\mu \delta\chi + \underbrace{\delta\chi^\dagger \pi^\dagger}_{\text{vanishes for unsymmetrized Lagrangian.}}\Big]$$

If Lagrangian is invariant (up to a divergence) under the transformation,

$$\delta\mathcal{L} = \partial_\mu V^\mu$$

Then, $0 = \partial_\mu J^\mu - \partial_\mu V^\mu = \partial_\mu(\underbrace{J^\mu - V^\mu}_{\text{Noether current}})$.

The associated charge is:

$$Q = \int d^3x \; J^0(\vec{x}, t) \qquad \text{where } \frac{dQ}{dt} = 0.$$