Energy/Momentum tensor

$$\mathcal{L} = i\chi^\dagger \bar{\sigma}^\mu \partial_\mu \chi - \frac{1}{2} m (\chi\chi + \chi^\dagger \chi^\dagger) \leftarrow \text{Majorana Mass}$$

$$= i\chi^\dagger_{\dot{a}} \bar{\sigma}^{\mu \dot{a} b} \partial_\mu \chi_b - \frac{1}{2} m (\chi^a \chi_a + \chi^\dagger_{\dot{a}} \chi^{\dagger \dot{a}})$$

Canonical Momenta

$$\pi^{\nu c} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \chi_d)} = i\chi^\dagger \bar{\sigma}^{\mu a b} \delta_\mu{}^\nu \delta_b{}^c$$

(remember derivative with respect to χ acts from right)

$$= i\chi^\dagger \bar{\sigma}^{\nu a c}$$

$$\pi^{\dagger \nu \dot{c}} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \chi^\dagger_{\dot{c}})} = 0$$

(since Lagrangian is not symmetrized)

Under translations, $x^\mu \to x^\mu + a^\mu$ $\quad a^\mu \equiv$ infinitesimal

$$\chi(x) \to \chi(x+a) \simeq \chi(x) + \underbrace{a_\nu \partial^\nu \chi(x)}_{\delta\chi}$$

$$\chi^\dagger(x) \to \chi^\dagger(x+a) \simeq \chi^\dagger(x) + \underbrace{a_\nu \partial^\nu \chi^\dagger(x)}_{\delta\chi^\dagger}$$

Lagrangian: $\mathcal{L}(x) \longrightarrow \mathcal{L}(x+a) = \mathcal{L}(x) + \underbrace{a_\nu \partial^\nu \mathcal{L}(x)}_{\delta\mathcal{L}} \Rightarrow V^\mu = a_\nu g^{\mu\nu} \mathcal{L}$

So, the associated Nöether current is

$$a_\nu T^{\mu\nu} = \pi^{\mu a} a_\nu \partial^\nu \chi_a + a_\nu \partial^\nu \chi^\dagger_{\dot{a}} \underset{0}{\underset{\shortparallel}{\pi^{\dagger \mu \dot{a}}}} - a_\nu g^{\mu\nu} \mathcal{L}$$

$$= a_\nu (\pi^\mu \partial^\nu \chi - g^{\mu\nu} \mathcal{L})$$

So $\boxed{T^{\mu\nu} = i\chi^\dagger \bar{\sigma}^\mu \partial^\nu \chi - g^{\mu\nu} \mathcal{L}}$, and the conserved charges are:

$$P^\mu = \int d^3x \; T^{0\mu} = \int d^3x \left(i\chi^\dagger \bar{\sigma}^0 \partial^\mu \chi - g^{0\mu} \mathcal{L} \right)$$

Angular momentum tensor

Under rotations, $x^\mu \longrightarrow \Lambda^\mu{}_\nu\, x^\nu$

$$\chi_a(x) \longrightarrow L_a{}^b(\Lambda)\,\chi_b(\Lambda^{-1}x) \qquad L_a{}^b = \delta_a{}^b - \frac{i}{2}\theta_{\mu\nu}(S_L^{\mu\nu})_a{}^b$$

$$\simeq \chi_a + \frac{\theta_{\mu\nu}}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu)\chi_a - \frac{i}{2}\theta_{\mu\nu}(S_L^{\mu\nu})_a{}^b\chi_b$$

Lagrangian: $\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1}x) = \mathcal{L}(x) + \frac{\theta_{\mu\nu}}{2}(x^\mu\partial^\nu - x^\nu\partial^\mu)\mathcal{L}$

$$\Rightarrow V^\mu = \frac{\theta_{\nu\rho}}{2}(x^\nu g^{\mu\rho} - x^\rho g^{\mu\nu})\mathcal{L}$$

So, the associated Nöether current is:

$$\frac{\theta_{\nu\rho}}{2} J^{\mu\nu\rho}(x) = \pi^\mu\left(\frac{\theta_{\nu\rho}}{2}(x^\nu\partial^\rho - x^\rho\partial^\nu) - \frac{i}{2}\theta_{\nu\rho}(S_L^{\nu\rho})\right)\chi$$

recall infinitesimal parameter $\epsilon = \frac{\theta_{\mu\nu}}{2}$

$$+ \left(\text{vanishing } \pi^{k\mu} \text{ term}\right) - \frac{\theta_{\nu\rho}}{2}(x^\nu g^{\mu\rho} - x^\rho g^{\mu\nu})\mathcal{L}$$

$$= \frac{\theta_{\nu\rho}}{2}\Big[x^\nu(\pi^\mu\partial^\rho\chi - g^{\mu\rho}\mathcal{L}) - x^\rho(\pi^\mu\partial^\nu\chi - g^{\mu\nu}\mathcal{L}) - i\pi^\mu(S_L^{\nu\rho})\chi\Big]$$

components of currents   Labels currents   (6 currents)

$$\boxed{J^{\mu\,\nu\rho} = \underbrace{x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}}_{\text{orbital contribution}} - \underbrace{i\pi^\mu(S_L^{\nu\rho})\chi}_{\text{spin contribution}}}$$

Antisymmetric under $\nu \longleftrightarrow \rho$, and is conserved: $\partial_\mu J^{\mu\nu\rho} = 0$.

Conserved charge / generator of rotations:

$$\hat{M}^{\mu\nu} = \int d^3x\; J^{0\,\mu\nu}(x)$$

$$= \int d^3x\left(x^\mu T^{0\nu} - x^\nu T^{0\mu} - i\pi^0(S_L^{\mu\nu})\chi\right)$$

$$= \int d^3x\left(2x^{[\mu}T^{|0|\nu]} - i\pi^0(S_L^{\mu\nu})\chi\right)$$

Look at second term in $J^{\mu\,\nu\rho}(x)$:

$$-i\pi^{\mu}(S_L^{\nu\rho})\chi = -i(i\chi^\dagger\bar\sigma^\mu)\frac{i}{4}(\sigma^\nu\bar\sigma^\rho - \sigma^\rho\bar\sigma^\nu)\chi$$

$$= \frac{i}{4}\chi^\dagger(\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho - \bar\sigma^\mu\sigma^\rho\bar\sigma^\nu)\chi$$

use identity: $\bar\sigma^\mu\sigma^\nu\bar\sigma^\rho = g^{\mu\nu}\bar\sigma^\rho - g^{\mu\rho}\bar\sigma^\nu + g^{\nu\rho}\bar\sigma^\mu - i\epsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma$

$$-i\pi^{\mu}(S_L^{\nu\rho})\chi^\dagger = \frac{i}{4}\chi^\dagger\Big[g^{\mu\nu}\bar\sigma^\rho - g^{\mu\rho}\bar\sigma^\nu + \cancel{g^{\nu\rho}\bar\sigma^\mu} - i\epsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma$$
$$- g^{\mu\rho}\bar\sigma^\nu + g^{\mu\nu}\bar\sigma^\rho - \cancel{g^{\rho\nu}\bar\sigma^\mu} + i\epsilon^{\mu\rho\nu\sigma}\bar\sigma_\sigma\Big]\chi$$

$$= \frac{i}{4}\chi^\dagger\big[2g^{\mu\nu}\bar\sigma^\rho - 2g^{\mu\rho}\bar\sigma^\nu - 2i\epsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma\big]\chi$$

$$= \frac{i}{2}\chi^\dagger\left(g^{\mu\nu}\bar\sigma^\rho - g^{\mu\rho}\bar\sigma^\nu - i\epsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma\right)\chi$$

for convenience, can remove bar by using $\chi^\dagger\bar\sigma\chi = -\chi\sigma\chi^\dagger$

$$= \frac{i}{2}\chi\left(g^{\mu\rho}\sigma^\nu - g^{\mu\nu}\sigma^\rho + i\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma\right)\chi^\dagger.$$

So, the angular momentum current density tensor is

$$\boxed{J^{\mu\,\nu\rho} = \underbrace{x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}}_{\text{orbital part}} + \underbrace{\frac{i}{2}\chi\left(g^{\mu\rho}\sigma^\nu - g^{\mu\nu}\sigma^\rho + i\epsilon^{\mu\nu\rho\sigma}\sigma_\sigma\right)\chi^\dagger}_{\text{spin part}}}$$

$$-\frac{i}{2}\chi^\dagger\left(g^{\mu\nu}\bar\sigma^\rho - g^{\mu\rho}\bar\sigma^\nu - i\epsilon^{\mu\nu\rho\sigma}\bar\sigma_\sigma\right)\chi$$

$$\hat M^{\mu\nu} = \int d^3x\, J^{0\,\mu\nu}(x)$$

$$= \int d^3x\left[x^\mu T^{0\nu} - x^\nu T^{0\mu} - \frac{i}{2}\chi^\dagger\left(g^{0\mu}\bar\sigma^\nu - g^{0\nu}\bar\sigma^\mu - i\epsilon^{0\mu\nu\rho}\bar\sigma_\rho\right)\chi\right]$$

$$= \int d^3x\left[\underbrace{i\chi^\dagger\bar\sigma^0\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\chi}_{\text{orbital part}} - \underbrace{\frac{i}{2}\chi^\dagger\left(g^{0\mu}\bar\sigma^\nu - g^{0\nu}\bar\sigma^\mu - i\epsilon^{0\mu\nu\rho}\bar\sigma_\rho\right)\chi}_{\text{spin part}}\right]$$