Transition to Dirac Spinor

Recall, when a mass term is added to Weyl spinor Lagrangian,

$$\mathcal{L} = \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi - \frac{1}{2} m_M (\chi\chi + \chi^\dagger\chi^\dagger)$$

the inherent chiral $U(1)$ symmetry is lost.

- This mass term is a Majorana mass term.

If instead, we had two (left-handed) non-interacting Weyl fermions, $\lambda^{(1)}$, $\lambda^{(2)}$, with the same mass term:

$\lambda^{(1)}$: C-even, P-even
$\lambda^{(2)}$: C-odd, P-even

$$\mathcal{L} = \lambda^{(1)\dagger} i\bar{\sigma}^\mu \partial_\mu \lambda^{(1)} + \lambda^{(2)\dagger} i\bar{\sigma}^\mu \partial_\mu \lambda^{(2)} - \frac{1}{2} m (\lambda^{(1)}\lambda^{(1)} + \lambda^{(1)\dagger}\lambda^{(1)\dagger}) - \frac{1}{2} m (\lambda^{(2)}\lambda^{(2)} + \lambda^{(2)\dagger}\lambda^{(2)\dagger})$$

there is an $SO(2)$ symmetry that mixes $\lambda^{(1)}$ & $\lambda^{(2)}$:

$$SO(2): \quad \begin{Bmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{Bmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{Bmatrix} \lambda^{(1)} \\ \lambda^{(2)} \end{Bmatrix} \qquad \begin{Bmatrix} \lambda^{(1)\dagger} \\ \lambda^{(2)\dagger} \end{Bmatrix} \text{ transforms in the same way.}$$

The $SO(2)$ symmetry can be recast as a $U(1)$ symmetry by writing:

inverse relations

$$\chi_\alpha = \frac{1}{\sqrt{2}}(\lambda^{(1)} + i\lambda^{(2)})_\alpha \qquad \lambda^{(1)}_\alpha = \frac{1}{\sqrt{2}}(\chi + \bar{\chi})_\alpha$$

$$\bar{\chi}_\alpha = \frac{1}{\sqrt{2}}(\lambda^{(1)} - i\lambda^{(2)})_\alpha \qquad \lambda^{(2)}_\alpha = \frac{-i}{\sqrt{2}}(\chi - \bar{\chi})_\alpha$$

(not quite dagger)

$$U(1): \quad \chi \to e^{i\theta}\chi \qquad \chi^\dagger \to e^{-i\theta}\chi^\dagger$$

$$\bar{\chi} \to e^{-i\theta}\bar{\chi} \qquad \bar{\chi}^\dagger \to e^{i\theta}\chi$$

with θ same as in $SO(2)$ transformation

The Lagrangian then becomes:

$$\mathcal{L} = \frac{1}{\sqrt{2}}(\chi^\dagger + \bar{\chi}^\dagger)\, i\bar{\sigma}^\mu\partial_\mu \frac{1}{\sqrt{2}}(\chi + \bar{\chi}) \;+\; \frac{i}{\sqrt{2}}(\chi^\dagger - \bar{\chi}^\dagger)\, i\bar{\sigma}^\mu\partial_\mu \left(\frac{-i}{\sqrt{2}}\right)(\chi - \bar{\chi})$$

$$- \frac{1}{2} m \left(\frac{1}{\sqrt{2}}\right)^2 (\chi + \bar{\chi})(\chi + \bar{\chi}) + \text{h.c.} \;-\; \frac{1}{2} m \left(\frac{-i}{\sqrt{2}}\right)^2 (\chi - \bar{\chi})(\chi - \bar{\chi}) + \text{h.c.}$$

$$= \frac{1}{2}\left(\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\bar{\chi} + \bar{\chi}^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \bar{\chi}^\dagger i\bar{\sigma}^\mu\partial_\mu\bar{\chi}\right)$$

$$+ \frac{1}{2}\left(\chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi - \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\bar{\chi} - \bar{\chi}^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \bar{\chi}^\dagger i\bar{\sigma}^\mu\partial_\mu\bar{\chi}\right)$$

$$- m\frac{1}{2}\left(\frac{1}{2}\right)\left(\chi\chi + \chi\bar{\chi} + \bar{\chi}\chi + \bar{\chi}\bar{\chi} + \text{h.c.}\right) + m\frac{1}{2}\left(\frac{1}{2}\right)\left(\chi\chi - \chi\bar{\chi} - \bar{\chi}\chi + \bar{\chi}\bar{\chi} + \text{h.c.}\right)$$

$\boxed{\text{MANY CANCELLATIONS}}$

notice: χ & $\bar{\chi}$ coupled.

$$= \chi^\dagger i\bar{\sigma}^\mu\partial_\mu\chi + \bar{\chi}^\dagger i\bar{\sigma}^\mu\partial_\mu\bar{\chi} - m(\chi\bar{\chi} + \chi^\dagger\bar{\chi}^\dagger)$$

The $U(1)_V$ symmetry is manifest:

$$\chi \to e^{i\theta}\chi \qquad \chi^\dagger \to e^{-i\theta}\chi^\dagger$$
$$\bar{\chi} \to e^{-i\theta}\bar{\chi} \qquad \bar{\chi}^\dagger \to e^{i\theta}\bar{\chi}^\dagger.$$

associated Noether current:

$$J^\mu_V = -\left(\chi^\dagger\bar{\sigma}^\mu\chi - \bar{\chi}^\dagger\bar{\sigma}^\mu\bar{\chi}\right)$$

Next: stack χ "left handed" and $\bar{\chi}^\dagger$ "right handed" Weyl spinors to form Dirac spinor.

Notice: we chose under parity

$$\lambda^{(1)} \xrightarrow{P} +i\underset{\substack{\uparrow \\ \left(\begin{smallmatrix}1 & \\ & 1\end{smallmatrix}\right) \\ \text{(id. matrix)}}}{\sigma^0} \lambda^{\dagger(1)} \quad \& \quad \lambda^{(2)} \xrightarrow{P} +i\sigma^0 \lambda^{\dagger(2)}$$

So that

$$\chi \xrightarrow{P} \frac{\sigma^0}{\sqrt{2}}\left(i\lambda^{(1)\dagger} - \lambda^{(2)\dagger}\right) = \frac{i\sigma^0}{\sqrt{2}}\left(\lambda^{(1)} - i\lambda^{(2)}\right)^\dagger = i\sigma^0 \bar{\chi}^\dagger$$

$$\bar{\chi} \xrightarrow{P} \frac{\sigma^0}{\sqrt{2}}\left(i\lambda^{(1)\dagger} + \lambda^{(2)\dagger}\right) = \frac{i\sigma^0}{\sqrt{2}}\left(\lambda^{(1)} + i\lambda^{(2)}\right)^\dagger = i\sigma^0 \chi^\dagger$$

Therefore if I define $\psi = \begin{pmatrix} \chi \\ \bar{\chi}^\dagger \end{pmatrix}$, under parity,

$$\psi \xrightarrow{P} \begin{pmatrix} i\sigma^0 \bar{\chi}^\dagger \\ i\sigma^0 \chi \end{pmatrix} = i\underbrace{\begin{pmatrix} 0 & \sigma^0 \\ \bar{\sigma}^0 & 0 \end{pmatrix}}_{\gamma^0}\begin{pmatrix} \chi \\ \bar{\chi}^\dagger \end{pmatrix} = i\gamma^0\psi \quad \text{(switches the two spinors)}$$

(Standard / Chiral basis)

ψ transforms under the $(\frac{1}{2}, 0)^{+i} \oplus (0, \frac{1}{2})^{+i}$ representation of the L. group, and as defined above, is a <u>parity positive-i Dirac spinor</u>

A <u>parity negative-i</u> Dirac spinor $(\frac{1}{2}, 0)^{-i} \oplus (0, \frac{1}{2})^{-i}$ can similarly be constructed.

<u>NOTE:</u> It is impossible to construct parity even/odd Dirac spinors $\psi \xrightarrow{P} \pm\gamma^0\psi$ from Weyl spinors.

Reason: Then particle and antiparticles have opposite intrinsic parity, and it would be <u>impossible</u> to satisfy the Majorana condition

Parity of charge conjugated spinors: (Weyl basis)

If $\psi \xrightarrow{P} \eta\gamma^0\psi$ then $\psi^c \xrightarrow{P} -\eta^*\gamma^0\psi^c$

$\bar{\psi} \xrightarrow{P} \eta^*\bar{\psi}\gamma^0$ $\quad\bar{\psi^c} \xrightarrow{P} -\eta\,\bar{\psi^c}\gamma^0$

<u>Proof:</u>

$\psi^c = i\gamma^2\gamma^0\bar{\psi}^*$

$\longrightarrow i\gamma^2\gamma^0(\gamma^0)^*\psi\,\eta^*$

$= -i(\gamma^0)\, i\gamma^2\gamma^0\psi\,\eta^*$

$= -i(\gamma^0)\,\psi^c\,\eta^*$

The Lagrangian for the Dirac spinor can be constructed by stacking:

$$\psi = \begin{pmatrix} \chi \\ \bar{\chi}^\dagger \end{pmatrix} : \quad \left(\tfrac{1}{2}, 0\right)^{+} \oplus \left(0, \tfrac{1}{2}\right)^{+}$$ under Lorentz transformations.

Let $\beta = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}$ so that $\bar{\psi} = \psi^\dagger \beta = \left(\bar{\chi} \quad \chi^\dagger\right)$

Write $\bar{\chi}^\dagger i\bar{\sigma}^\mu \partial_\mu \bar{\chi} = -\partial_\mu \bar{\chi}\, i\sigma^\mu \bar{\chi}^\dagger \xrightarrow{\text{IBP}} \bar{\chi}\, i\sigma^\mu \partial_\mu \bar{\chi}^\dagger$

So that $\mathcal{L} = \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + \bar{\chi}\, i\sigma^\mu \cdot \partial_\mu \bar{\chi}^\dagger - m\,\chi\bar{\chi} - m\,\chi^\dagger\bar{\chi}^\dagger$

$$= \left(\bar{\chi} \quad \chi^\dagger\right) \underbrace{i \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix}}_{\text{Define} \,=:\, \gamma^\mu} \partial_\mu \begin{pmatrix} \chi \\ \bar{\chi}^\dagger \end{pmatrix} - m \left(\bar{\chi} \quad \chi^\dagger\right) \begin{pmatrix} \chi^\dagger \\ \bar{\chi}^\dagger \end{pmatrix}$$

$$= \bar{\psi}\, i\gamma^\mu \partial_\mu \psi - m \bar{\psi}\psi$$

$$= \bar{\psi}\left(i\not{\partial} - m\right)\psi$$