Noether's theorem for Dirac fermions:

Lagrangian: $\mathcal{L} = \mathcal{L}[\psi, \partial_\mu\psi, \bar\psi, \partial_\mu\bar\psi]$

Transformation law: $\psi \to \psi + \delta\psi \qquad \bar\psi \to \bar\psi + \delta\bar\psi$

Lagrangian transforms: $\mathcal{L}[\psi] \to \mathcal{L}[\psi] + \delta\mathcal{L}$

where

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi) + \delta\bar\psi\frac{\partial\mathcal{L}}{\partial\bar\psi} + \delta(\partial_\mu\bar\psi)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)} \qquad (*)$$

Equations of motion:

$$\frac{\partial\mathcal{L}}{\partial\psi} = \partial_\mu \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}}_{\text{abbr: } \pi^\mu} \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial\bar\psi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar\psi)}$$

Then $(*)$ becomes:

$$\delta\mathcal{L} = (\partial_\mu\pi^\mu)\delta\psi + \pi^\mu(\partial_\mu\delta\psi) + \delta\bar\psi(\partial_\mu\bar\pi^\mu) + \partial_\mu(\delta\bar\psi)\bar\pi^\mu$$

$$= \partial_\mu(\pi^\mu\delta\psi + \underbrace{\delta\bar\psi\,\bar\pi^\mu}_{\substack{\text{vanishes for unsymmetrized}\\ \text{Lagrangian.}}})$$

If Lagrangian is invariant (up to a divergence) under transformation,

$$\delta\mathcal{L} = \partial_\mu V^\mu$$

then,

$$0 = \partial_\mu\underbrace{(\pi^\mu\delta\psi + \delta\bar\psi\,\bar\pi^\mu - V^\mu)}_{\text{Noether current } J^\mu}$$

with associated conserved charge:

$$Q = \int d^3x\; J^0(\vec{x}, t) \qquad \dot{Q} = 0.$$

Examples: Vector transformation $\psi \to e^{-i\alpha}\psi \Rightarrow \delta\psi = -i\alpha\psi$

$$\delta\mathcal{L} = \alpha\,\partial_\mu\underbrace{(\bar\psi\gamma^\mu\psi)}_{J_V^\mu} = 0$$

Axial transformation $\psi \to e^{-i\alpha\gamma_5}\psi \Rightarrow \delta\psi = -i\alpha\gamma_5\psi$

$$\delta\mathcal{L} = \alpha\,\partial_\mu\underbrace{(\bar\psi\gamma^\mu\gamma_5\psi)}_{J_A^\mu} = 0$$

Examples:

4-component Dirac | Two component Weyl

Vector

$U(1)_V$: $\psi \to e^{-i\alpha}\psi$ $\qquad$ $\psi = \begin{pmatrix}\chi \\ \bar{\chi}^\dagger\end{pmatrix} \longrightarrow \begin{pmatrix} e^{-i\alpha} & \\ & e^{-i\alpha}\end{pmatrix}\begin{pmatrix}\chi \\ \bar{\chi}^\dagger\end{pmatrix}$

$$\delta\mathcal{L} = \partial_\mu \underbrace{(\bar{\psi}\gamma^\mu\psi)}_{J^\mu_V}$$

$$\delta\mathcal{L} = \partial_\mu\Big[\overset{(\text{Left})}{\chi^\dagger\bar{\sigma}^\mu\chi} \overset{+}{+} \overset{(\text{Right})}{\bar{\chi}\sigma^\mu\bar{\chi}^\dagger}\Big]$$
$$= \partial_\mu\left[\chi^\dagger\bar{\sigma}^\mu\chi - \bar{\chi}^\dagger\bar{\sigma}^\mu\bar{\chi}\right]$$

Axial

$U(1)_A$: $\psi \to e^{-i\alpha\gamma_5}\psi$ $\qquad$ $\psi = \begin{pmatrix}\chi \\ \bar{\chi}^\dagger\end{pmatrix} \longrightarrow \begin{pmatrix} e^{+i\alpha} & \\ & e^{-i\alpha}\end{pmatrix}\begin{pmatrix}\chi \\ \bar{\chi}^\dagger\end{pmatrix}$

$$\delta\mathcal{L} = \partial_\mu \underbrace{(\bar{\psi}\gamma^\mu\gamma_5\psi)}_{J^\mu_A}$$

$$\delta\mathcal{L} = \partial_\mu\Big(\overset{(-\text{Left})}{-\chi^\dagger\bar{\sigma}^\mu\chi} + \overset{(+\text{Right})}{\bar{\chi}\sigma^\mu\bar{\chi}^\dagger}\Big)$$