Orthonormalization

Non-covariant: (fixed-axis quantization in standard rep.)

$$u^\dagger_{s'}(\vec{p})\, u_s(\vec{p}) = (E+m)\left[\xi^\dagger_{s'}\xi_s + \xi^\dagger_{s'} \overbrace{\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\,\frac{\vec{p}\cdot\vec{\sigma}}{E+m}}^{p^2} \xi_s\right]$$

$$= (E+m)\left[1 + \frac{\vec{p}^{\,2}}{(E+m)^2}\right]\delta_{s's} = 2E\,\delta_{s's}$$

$$v^\dagger_{s'}(\vec{p})\, v_s(\vec{p}) = 2E\,\delta_{s's} \qquad \text{[same]}$$

$$u^\dagger_{s'}(-\vec{p})\, v_s(\vec{p}) = \sqrt{E+m}\begin{pmatrix}\xi^\dagger_{s'} & \dfrac{-\vec{p}\cdot\vec{\sigma}}{E+m}\xi^\dagger_{s'}\end{pmatrix}\begin{pmatrix}\dfrac{\vec{p}\cdot\vec{\sigma}}{E+m}\zeta_s \\ \zeta_s\end{pmatrix}\sqrt{E+m}$$

$$= (E+m)\left[\xi^\dagger_{s'}\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\zeta_s + \xi^\dagger_{s'}\frac{-\vec{p}\cdot\vec{\sigma}}{E+m}\zeta_s\right] = 0.$$

$$v^\dagger_{s'}(-\vec{p})\, u_s(\vec{p}) = 0. \qquad \text{[same]}$$

Covariant:

$$\bar{u}_{s'}(\vec{p})\, u_s(\vec{p}) = 2m\,\delta_{s's} \qquad \bar{u}_{s'}(\vec{p})\, v_s(\vec{p}) = 0$$

$$\bar{v}_{s'}(\vec{p})\, v_s(\vec{p}) = -2m\,\delta_{s's} \qquad \bar{v}_{s'}(\vec{p})\, u_s(\vec{p}) = 0$$

Spinor equation of motion

$$\not{p}\, u_s(\vec{p}) = m\, u_s(\vec{p}) \qquad \not{p}\, v_s(\vec{p}) = (-m)\, v_s(\vec{p})$$

$$\bar{u}_s(\vec{p})\,\not{p} = \bar{u}_s(\vec{p})\, m \qquad \bar{v}_s(\vec{p})\,\not{p} = \bar{v}_s(\vec{p})\,(-m)$$

Completeness relations

Positive frequency

$$\sum_{s=\pm 1/2} u^{[s]}_{\text{Dirac}}(\vec{p})\ \bar{u}^{[s]}_{\text{Dirac}}(\vec{p})$$

$$= \sum_s \begin{pmatrix} u_\alpha^{[s]}(\vec{p}) \\ v^{\dagger\dot{\alpha}[s]}(\vec{p}) \end{pmatrix} \otimes \left(v^{\beta[s]}(\vec{p}), \quad u^{\dagger[s]}_{\dot{\beta}}(\vec{p}) \right)$$

$$= \sum_s \begin{pmatrix} u_\alpha v^\beta & u_\alpha u^\dagger_{\dot{\beta}} \\ v^{\dagger\dot{\alpha}} v^\beta & v^{\dagger\dot{\alpha}} u^\dagger_{\dot{\beta}} \end{pmatrix}$$

$$= \begin{pmatrix} m\,\delta_\alpha{}^\beta & p\cdot\sigma_{\alpha\dot{\beta}} \\ p\cdot\bar{\sigma}^{\dot{\alpha}\beta} & m\,\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} = \not{p} + m$$

Negative Frequency

$$\sum_{s=\pm 1/2} v^{[s]}_{\text{Dirac}}(\vec{p})\ \bar{v}^{[s]}_{\text{Dirac}}(\vec{p})$$

$$= \sum_s \begin{pmatrix} v_\alpha^{[s]}(\vec{p}) \\ u^{\dagger\dot{\alpha}[s]}(\vec{p}) \end{pmatrix} \otimes \left(u^{\beta[s]}(\vec{p}), \quad v^{\dagger[s]}_{\dot{\beta}}(\vec{p}) \right)$$

$$= \sum_s \begin{pmatrix} v_\alpha u^\beta & v_\alpha v^\dagger_{\dot{\beta}} \\ u^{\dagger\dot{\alpha}} u^\beta & u^{\dagger\dot{\alpha}} v^\dagger_{\dot{\beta}} \end{pmatrix}$$

$$= \begin{pmatrix} -m\,\delta_\alpha{}^\beta & p\cdot\sigma_{\alpha\dot{\beta}} \\ p\cdot\bar{\sigma}^{\dot{\alpha}\beta} & -m\,\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix} = \not{p} - m.$$

For completeness relations for unusual combinations, see notes on Majorana fermions.

Positive/negative frequency projector

$$\Lambda_\pm(p) = \frac{1}{2m}\left(\pm \not{p} + m\right)$$

$$= \frac{1}{2m}\begin{pmatrix} m\delta_\alpha{}^\beta & \pm p.\sigma_{\alpha\dot\beta} \\ \pm p.\bar\sigma^{\dot\alpha\beta} & m\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}$$

Spin Projector

$$\Sigma^{[m_s]}(s^\mu) = \frac{1}{2}\left(1 + 2\underbrace{m_s}_{s=\pm 1/2}\gamma_5 \not{s}\right)$$

$$= \frac{1}{2}\begin{pmatrix} \delta_\alpha{}^\beta & -2m_s\, S\cdot\sigma_{\alpha\dot\beta} \\ 2m_s\, S\cdot\bar\sigma^{\dot\alpha\beta} & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}$$

$S^\mu \equiv$ boosted quantization axis. $(S_\mu S^\mu = -1)$

Identities:

$$u^{[s]}(\vec p)\,\bar u^{[s]}(\vec p) = 2m\Lambda_+(\vec p)\,\Sigma^{[s]}(S^\mu)$$

i.e. (spin proj. op) $\times$ (result of sum over all spins)

$$v^{[s]}(\vec p)\,\bar v^{[s]}(\vec p) = -2m\Lambda_-(\vec p)\,\Sigma^{[s]}(S^\mu)$$

$$\Sigma^{[s]}(S^\mu)\,\Sigma^{[s]}(S^\mu) = \Sigma^{[s]}(S^\mu)$$

$$\Lambda_\pm(\vec p)\,\Lambda_\pm(\vec p) = \Lambda_\pm(\vec p)$$

Projectors commute with each other:

$$[\Sigma, \Lambda] = 0 \qquad \text{(act on distinct subspaces)}$$

Action on u & v spinors:

$$\Sigma^{[s]} u^{[s]}(\vec p) = u^{[s]}(\vec p) \qquad \Sigma^{[s]} v^{[s]}(\vec p) = v^{[s]}(\vec p)$$

$$\Sigma^{[s]} u^{[-s]}(\vec p) = 0 \qquad \Sigma^{[s]} v^{[-s]}(\vec p) = 0$$