Hiren H. Patel
hhpatel.net

(Active)
L. Transformation of Dirac Spinors

(spacetime arguments suppressed)

Write L. transf. in terms of 2-comp. spinors

$$\psi = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dagger\dot\alpha} \end{pmatrix} \quad \text{with:} \quad \begin{aligned} \chi_\alpha &\longrightarrow \left[e^{-\frac{i}{2}\theta_{\mu\nu}(S_L^{\mu\nu})}\right]_\alpha{}^\beta \chi_\beta \\ \bar{\chi}^{\dagger\dot\alpha} &\longrightarrow \left[e^{-\frac{i}{2}\theta_{\mu\nu}(S_R^{\mu\nu})}\right]^{\dot\alpha}{}_{\dot\beta}\, \bar{\chi}^{\dagger\dot\beta} \end{aligned}$$

$$\therefore\ \psi \longrightarrow \begin{pmatrix} e^{-\frac{i}{2}\theta_{\mu\nu}(S_L^{\mu\nu})_\alpha{}^\beta} & \\ & e^{-\frac{i}{2}\theta_{\mu\nu}(S_R^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}} \end{pmatrix} \begin{pmatrix} \chi_\beta \\ \bar{\chi}^{\dagger\dot\beta} \end{pmatrix}$$

or

$$\begin{pmatrix} \psi \end{pmatrix} \longrightarrow \left[e^{-\frac{i}{2}\theta_{\mu\nu}(S_D^{\mu\nu})} \right] \begin{pmatrix} \psi \end{pmatrix}$$

Dirac spin matrix: (chiral basis)

$$S_D^{\mu\nu} = \begin{pmatrix} (S_L^{\mu\nu})_\alpha{}^\beta & \\ & (S_R^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} = \begin{pmatrix} \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu) & \\ & \frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu) \end{pmatrix}$$

$$= \frac{i}{4}\left[\begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix} \begin{pmatrix} & \sigma^\nu \\ \bar\sigma^\nu & \end{pmatrix} - \begin{pmatrix} & \sigma^\nu \\ \bar\sigma^\nu & \end{pmatrix} \begin{pmatrix} & \sigma^\mu \\ \bar\sigma^\mu & \end{pmatrix} \right]$$

$$= \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}\Sigma^{\mu\nu}$$

common abbr. in literature
$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$
$$\Sigma^{\mu\nu} = [\gamma^\mu, \gamma^\nu]$$

Explicitly,

$$S_D^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \begin{pmatrix} 0 & \begin{pmatrix} i\sigma^1 & \\ & -i\sigma^1 \end{pmatrix} & \begin{pmatrix} i\sigma^2 & \\ & -i\sigma^2 \end{pmatrix} & \begin{pmatrix} i\sigma^3 & \\ & -i\sigma^3 \end{pmatrix} \\ & 0 & \begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix} & \begin{pmatrix} -\sigma^2 & \\ & -\sigma^2 \end{pmatrix} \\ & & 0 & \begin{pmatrix} \sigma^1 & \\ & \sigma^1 \end{pmatrix} \\ & & & 0 \end{pmatrix}$$

Boost generators (chiral basis)

Rotation generators

Hiren H. Patel
hhpatel.net

Dirac spin matrices (generator of Lorentz transformations)

Definition:

$$S_D^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right] = \frac{1}{2}\sigma^{\mu\nu} \quad \text{or} = \frac{i}{4}\Sigma^{\mu\nu}$$

Chiral Representation

$$(S_D)^{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & \begin{pmatrix} i\sigma^1 & \\ & i\sigma^1 \end{pmatrix} & \begin{pmatrix} i\sigma^2 & \\ & i\sigma^2 \end{pmatrix} & \begin{pmatrix} i\sigma^3 & \\ & i\sigma^3 \end{pmatrix} \\ & 0 & \begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix} & \begin{pmatrix} -\sigma^2 & \\ & -\sigma^2 \end{pmatrix} \\ & & 0 & \begin{pmatrix} \sigma^1 & \\ & \sigma^1 \end{pmatrix} \\ & & & 0 \end{pmatrix}$$

← generator of boosts (first row)

generator of rotations (remaining rows)

Standard Representation

$$(S_D)^{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & \begin{pmatrix} & i\sigma^1 \\ i\sigma^1 & \end{pmatrix} & \begin{pmatrix} & i\sigma^2 \\ i\sigma^2 & \end{pmatrix} & \begin{pmatrix} & i\sigma^3 \\ i\sigma^3 & \end{pmatrix} \\ & 0 & \begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix} & \begin{pmatrix} -\sigma^2 & \\ & -\sigma^2 \end{pmatrix} \\ & & 0 & \begin{pmatrix} \sigma^1 & \\ & \sigma^1 \end{pmatrix} \\ & & & 0 \end{pmatrix}$$

Hiren H. Patel
hhpatel.net

Active
Lorentz transformation matrix for Dirac spinors

$$\psi \to R(\Lambda)\,\psi$$

and
$$R(\Lambda) = \left(e^{-\frac{i}{2}\theta_{\mu\nu}\,(\overbrace{S_D^{\mu\nu}}^{\frac{i}{2}\sigma^{\mu\nu}})} \right)$$

ROTATIONS

$$R(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{S}} = e^{-i\vec{\theta}\cdot\begin{pmatrix} \vec{\sigma}/2 & \\ & \vec{\sigma}/2 \end{pmatrix}}$$

If $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ is written $\theta\cdot\hat{\theta}$, where $\theta = \sqrt{\theta_1^2 + \theta_2^2 + \theta_3^2}$

$\hat{\theta} = \vec{\theta}/\theta$ = axis of rotation.

$$R(\vec{\theta}) = \mathbb{1}_{4\times4}\cos\frac{\theta}{2} - i\begin{pmatrix} \vec{\sigma} & \\ & \vec{\sigma} \end{pmatrix}\cdot\hat{\theta}\,\sin\frac{\theta}{2}$$

leads to phase change under 2π rotation.

valid in both Chiral and standard basis.

BOOSTS

$$\eta = \tanh^{-1}(v/c)$$

In chiral rep,

$$R(\vec{\eta}) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cosh(\eta/2) - \frac{\vec{\eta}}{\eta}\cdot\vec{\sigma}\,\sinh(\eta/2) & \\ & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\cosh(\eta/2) + \frac{\vec{\eta}}{\eta}\cdot\vec{\sigma}\,\sinh(\eta/2) \end{pmatrix}$$ (Diagonal)

Transition to standard/Dirac rep:

$$R(\vec{\eta})_{stand.} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & \mathbb{1} \\ -\mathbb{1} & \mathbb{1} \end{pmatrix} R(\vec{\eta})_{ch.}\,\frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1} & -\mathbb{1} \\ \mathbb{1} & \mathbb{1} \end{pmatrix}$$

$$= \mathbb{1}_{4\times4}\cosh\frac{\eta}{2} + \hat{\eta}\cdot\begin{pmatrix} & \vec{\sigma} \\ \vec{\sigma} & \end{pmatrix}\sinh\eta/2$$

$$= \frac{1}{\sqrt{2m(E+m)}}\left[\mathbb{1}_{4\times4} + \vec{p}\cdot\begin{pmatrix} & \vec{\sigma} \\ \vec{\sigma} & \end{pmatrix}\right]$$