Conjugation identities

– valid for standard and chiral representations

Gamma-0 conjugation:

Close to C

$$\gamma^0 \gamma^\mu \gamma^0 = (\gamma^\mu)^\dagger = \gamma_\mu$$

Proof:

$\mu = 0: \quad \overbrace{\gamma^0\gamma^0}^{1}\gamma^0 = \gamma^0 = (\gamma^0)^\dagger$

$\mu = 1: \quad \gamma^0 \underbrace{\gamma^1 \gamma^0}_{\ominus} = -\gamma^1 = (\gamma^1)^\dagger$

$\mu = 2: \quad \gamma^0 \gamma^2 \gamma^0 = -\gamma^2 = (\gamma^2)^\dagger$

$\mu = 3: \quad \gamma^0 \gamma^3 \gamma^0 = -\gamma^3 = (\gamma^3)^\dagger$

Together, if $C = i\gamma^2\gamma^0$

$i\gamma^0\gamma^2 \;\gamma^\mu\; i\gamma^2\gamma^0$

$= -\gamma^0(\gamma^\mu)^*\gamma^0$

$= -(\gamma^\mu)^T$

Gamma-2 conjugation:

close to P

$$\gamma^2 \gamma^\mu \gamma^2 = (\gamma^\mu)^*$$

Proof:

$\mu = 0: \quad \gamma^2\gamma^0\gamma^2 = -\overbrace{\gamma^2\gamma^2}^{-1}\gamma^0 = +\gamma^0 = (\gamma^0)^*$

$\mu = 1: \quad \gamma^2\gamma^1\gamma^2 = -\gamma^2\gamma^2\gamma^1 = +\gamma^1 = (\gamma^1)^*$

$\mu = 2: \quad \gamma^2\gamma^2\gamma^2 = \quad = -\gamma^2 = (\gamma^2)^*$

$\mu = 3: \quad \gamma^2\gamma^3\gamma^2 = -\gamma^2\gamma^2\gamma^3 = +\gamma^3 = (\gamma^3)^*$

Gamma-1 conj.

$\mu = 0: \quad \gamma^1\gamma^0\gamma^1 = -\gamma^1\gamma^1\gamma^0 = \gamma^0$

$\mu = 1: \quad \gamma^1 \underbrace{\gamma^1\gamma^1}_{-1} = \quad = -\gamma^1$

$\mu = 2: \quad \gamma^1\gamma^2\gamma^1 = -\gamma^1\gamma^1\gamma^2 = \gamma^2$

$\mu = 3: \quad \gamma^1\gamma^3\gamma^1 = \quad = \gamma^3$

Gamma-3 conj.

$\mu = 0: \quad \gamma^3 \underbrace{\gamma^0\gamma^3}_{\ominus} = -\gamma^3\gamma^3\gamma^0 = +\gamma^0$

$\mu = 1: \quad \gamma^3 \underbrace{\gamma^1\gamma^3}_{\ominus} = \quad = +\gamma^1$

$\mu = 2: \quad \gamma^3\gamma^2\gamma^3 = \quad = +\gamma^2$

$\mu = 3: \quad \gamma^3\gamma^3\gamma^3 = \quad = -\gamma^3$

together:

Gamma-1 Gamma-3 conjugation

Close to T

$$\underbrace{\gamma^1\gamma^3}_{T} (\gamma^\mu) \underbrace{\gamma^3\gamma^1}_{T^{-1}} = (\gamma^\mu)^T = \gamma^3\gamma^1(\gamma^\mu)\gamma^1\gamma^3$$