

Algebra with  $\sigma^{\mu\nu}$

Definition:  $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \Rightarrow \gamma^\mu\gamma^\nu = g^{\mu\nu}\mathbb{1} - i\sigma^{\mu\nu} \quad \sigma^{\mu\nu} = -i g^{\mu\nu}\mathbb{1} + i\gamma^\mu\gamma^\nu$

$$[\sigma^{\mu\nu}, \gamma^\rho] = \sigma^{\mu\nu}\gamma^\rho - \gamma^\rho\sigma^{\mu\nu}$$

$$\begin{aligned} \sigma^{\mu\nu}\gamma^\rho &= \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\rho \\ &= \frac{i}{2}(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho) \\ &= \frac{i}{2}(-\gamma^\mu\gamma^\rho\gamma^\nu + 2\gamma^\mu\gamma^\nu\gamma^\rho + \gamma^\nu\gamma^\rho\gamma^\mu - 2\gamma^\nu g^{\mu\rho}) \\ &= \frac{i}{2}(+\gamma^\rho\gamma^\mu\gamma^\nu - 2g^{\mu\rho}\gamma^\nu + 2\gamma^\mu\gamma^\nu\gamma^\rho \\ &\quad - \gamma^\rho\gamma^\nu\gamma^\mu + 2g^{\nu\rho}\gamma^\mu - 2\gamma^\nu g^{\mu\rho}) \end{aligned}$$

$$= \frac{i}{2}[\gamma^\rho(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) - 4\gamma^\nu g^{\mu\rho} + 4\gamma^\mu g^{\nu\rho}]$$

$$= \gamma^\rho\sigma^{\mu\nu} + 2i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})$$

$$\therefore [\sigma^{\mu\nu}, \gamma^\rho] = 2i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho})$$

$$\{\sigma^{\mu\nu}\gamma_5, \gamma^\rho\} = \sigma^{\mu\nu}\gamma_5\gamma^\rho + \gamma^\rho\sigma^{\mu\nu}\gamma_5$$

$$\begin{aligned} \sigma^{\mu\nu}\gamma_5\gamma^\rho &= \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_5\gamma^\rho \\ &= \frac{-i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma^\rho\gamma_5 \\ &= (-\gamma^\rho\sigma^{\mu\nu} - 2i(\gamma^\mu g^{\nu\rho} - \gamma^\nu g^{\mu\rho}))\gamma_5 \end{aligned}$$

$\therefore$

$$\{\sigma^{\mu\nu}\gamma_5, \gamma^\rho\} = 2i(g^{\nu\rho}\gamma^\mu\gamma_5 - g^{\mu\rho}\gamma^\nu\gamma_5)$$

$\sigma^{\mu\nu}$  is "Self dual":

$$\sigma^{\mu\nu}\gamma_5 = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \Rightarrow \epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = -2i\sigma^{\mu\nu}\gamma_5$$

✓ checked with  $\mu=0, \nu=1$

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = \sigma^{\mu\nu} \sigma^{\rho\sigma} - \sigma^{\rho\sigma} \sigma^{\mu\nu}$$

$$= \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \sigma^{\rho\sigma} - \sigma^{\rho\sigma} \sigma^{\mu\nu}$$

$$= \frac{i}{2} \overbrace{\gamma^\mu \gamma^\nu} \sigma^{\rho\sigma} - \frac{i}{2} \overbrace{\gamma^\nu \gamma^\mu} \sigma^{\rho\sigma} - \sigma^{\rho\sigma} \sigma^{\mu\nu}$$

$$= \frac{i}{2} \overbrace{\gamma^\mu} \sigma^{\rho\sigma} \gamma^\nu - \frac{i}{2} \overbrace{\gamma^\nu} \sigma^{\rho\sigma} \gamma^\mu - \sigma^{\rho\sigma} \sigma^{\mu\nu}$$

$$+ \frac{i}{2} \gamma^\mu [\gamma^\nu, \sigma^{\rho\sigma}] - \frac{i}{2} \gamma^\nu [\gamma^\mu, \sigma^{\rho\sigma}]$$

$$= \frac{i}{2} \cancel{\sigma^{\rho\sigma}} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) - \cancel{\sigma^{\rho\sigma}} \sigma^{\mu\nu}$$

$$+ \frac{i}{2} [\gamma^\mu, \sigma^{\rho\sigma}] \gamma^\nu - \frac{i}{2} [\gamma^\nu, \sigma^{\rho\sigma}] \gamma^\mu$$

$$+ \frac{i}{2} \gamma^\mu [\gamma^\nu, \sigma^{\rho\sigma}] - \frac{i}{2} \gamma^\nu [\gamma^\mu, \sigma^{\rho\sigma}]$$

$$= \frac{i}{2} (-2i) (\gamma^\rho \underline{g^{\sigma\mu}} - \gamma^\sigma \underline{g^{\rho\mu}}) \gamma^\nu - \frac{i}{2} (-2i) (\gamma^\rho \underline{g^{\sigma\nu}} - \gamma^\sigma \underline{g^{\rho\nu}}) \gamma^\mu$$

$$+ \frac{i}{2} (-2i) \gamma^\mu (\gamma^\rho \underline{g^{\sigma\nu}} - \gamma^\sigma \underline{g^{\rho\nu}}) - \frac{i}{2} (-2i) \gamma^\nu (\gamma^\rho \underline{g^{\sigma\mu}} - \gamma^\sigma \underline{g^{\rho\mu}})$$

organize by  $g^{\alpha\beta}$

$$= 2i g^{\rho\mu} \frac{i}{2} (\gamma^\sigma \gamma^\nu - \gamma^\nu \gamma^\sigma) - 2i g^{\rho\nu} \frac{i}{2} (\gamma^\sigma \gamma^\mu - \gamma^\mu \gamma^\sigma)$$

$$- 2i g^{\sigma\mu} \frac{i}{2} (\gamma^\rho \gamma^\nu - \gamma^\nu \gamma^\rho) + 2i g^{\sigma\nu} \frac{i}{2} (\gamma^\rho \gamma^\mu - \gamma^\mu \gamma^\rho)$$

$$= 2i (g^{\rho\mu} \sigma^{\sigma\nu} - g^{\rho\nu} \sigma^{\sigma\mu} - g^{\sigma\mu} \sigma^{\rho\nu} + g^{\sigma\nu} \sigma^{\rho\mu})$$

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = -2i (g^{\mu\rho} \sigma^{\nu\sigma} - g^{\nu\rho} \sigma^{\mu\sigma} - g^{\mu\sigma} \sigma^{\nu\rho} + g^{\nu\sigma} \sigma^{\mu\rho})$$