$[\dots]_{12} = \bar{w}(p_1)\dots w(p_2)$

Identities for products of spinor chains

Contract $[3\times3]_{\pm\pm}$ (b) with $p_{i\beta}\, p_{j\rho}$
(Sirlin identity)

$$\gamma^\nu \not p_j \gamma^\mu P_\pm \otimes \gamma_\mu \not p_i \gamma_\nu P_\pm = 4 \not p_i P_\pm \otimes \not p_j P_\pm$$

w = u-spinor or v-spinor

Sandwich between $\bar w(p_1)[\dots]w(p_2)\ \bar w(p_3)[\dots]w(p_4)$ $\qquad \not p\, w(p) = \pm m\, w(p)$

$$[\gamma^\nu \not p_j\gamma^\mu P_\pm]_{12}[\gamma_\mu \not p_i\gamma_\nu P_\pm]_{34} = 4[\not p_i P_\pm]_{12}[\not p_j P_\pm]_{34}$$

Useful identities are obtained if both $\not p$'s are anticommuted towards γ^μ (or γ^ν).

① $p_j = p_1$, $p_i = p_4$

LHS $= [\gamma^\nu\not p_1\gamma^\mu P_\pm]_{12}[\gamma_\mu\not p_4\gamma_\nu P_\pm]_{34}$

$$= [(-\not p_1\gamma^\nu + 2p_1^\nu)\gamma^\mu P_\pm]_{12}[\gamma_\mu(-\gamma_\nu P_\mp \not p_4 + 2p_{4\nu}P_\pm)]_{34}$$

$$= [(-\mathrm{sgn}_1 m_1\gamma^\nu + 2p_1^\nu)\gamma^\mu P_\pm]_{12}[\gamma_\mu(-\gamma_\nu P_\mp \mathrm{sgn}_4\, m_4 + 2p_{4\nu}P_\pm)]_{34}$$

$$= \mathrm{sgn}_1 m_1\, \mathrm{sgn}_4 m_4 [\gamma^\nu\gamma^\mu P_\pm]_{12}[\gamma_\mu\gamma_\nu P_\mp]_{34}$$
$$- \mathrm{sgn}_1 m_1 [\gamma^\nu\gamma^\mu P_\pm]_{12}[\gamma_\mu P_\pm]_{34}\, 2p_{4\nu}$$
$$- \mathrm{sgn}_4 m_4\, 2p_1^\nu[\gamma^\mu P_\pm]_{12}[\gamma_\mu\gamma_\nu P_\mp]_{34}$$
$$+ 4 p_1\cdot p_4[\gamma^\mu P_\pm]_{12}[\gamma_\mu P_\pm]_{34}.$$

contract and resolve $\gamma^\mu\gamma^\nu \sim \mathbb{1} + \sigma^{\mu\nu}$.
$\mathcal{O}(m_1 m_4)$ parts cancel.

use computer algebra.

$$= -2i\,\mathrm{sgn}_1 m_1[\sigma^{\mu\nu}p_{4\nu}\hat P_\pm]_{12}[\gamma_\mu\hat P_\pm]_{34}$$
$$+2i\,\mathrm{sgn}_4 m_4[\gamma_\mu\hat P_\pm]_{12}[\sigma^{\mu\nu}p_{1\nu}\hat P_\mp]_{34}$$
$$+4p_1\cdot p_4[\gamma^\mu P_\pm]_{12}[\gamma_\mu P_\pm]_{34} \underset{\text{RHS}}{=} 4[\not p_4\hat P_\pm]_{12}[\not p_1\hat P_\pm]_{34} \qquad ①$$

② $p_j = p_2\,, \quad p_i = p_3$

$$\text{LHS} = \left[\gamma^\nu \not{p}_2 \gamma^\mu \hat{P}_\pm\right]_{12} \left[\gamma_\mu \not{p}_3 \gamma_\nu \hat{P}_\pm\right]_{34}$$

$$= \left[\gamma^\nu\left(-\gamma^\mu \hat{P}_\mp \not{p}_2 + 2p_2^\mu \hat{P}_\pm\right)\right]_{12} \left[\left(-\not{p}_3 \gamma_\mu + 2p_{3\mu}\right)\gamma_\nu \hat{P}_\pm\right]_{34}$$

$$= \left[\gamma^\nu\left(-\operatorname{sgn}_2 m_2 \gamma^\mu \hat{P}_\mp + 2p_2^\mu \hat{P}_\pm\right)\right]_{12} \left[\left(-\operatorname{sgn}_3 m_3 \gamma_\mu + 2p_{3\mu}\right)\gamma_\nu \hat{P}_\pm\right]_{34}$$

$$= 2i \operatorname{sgn}_2 m_2 \left[\sigma^{\mu\nu} p_{3\nu} \hat{P}_\mp\right]_{12} \left[\gamma_\mu \hat{P}_\pm\right]_{34}$$

$$- 2i \operatorname{sgn} m_3 \left[\gamma_\mu \hat{P}_\pm\right]_{12} \left[\sigma^{\mu\nu} p_{2\nu} \hat{P}_\pm\right]_{34}$$

$$+ 4 p_2 \cdot p_3 \left[\gamma_\mu \hat{P}_\pm\right]_{12} \left[\gamma^\mu \hat{P}_\pm\right]_{34} \,. \underset{\text{RHS}}{=} 4 \left[\not{p}_3 \hat{P}_\pm\right]_{12} \left[\not{p}_2 P_\pm\right]_{34} \qquad ②$$

Next, contract $[3\times 3]_{\pm\mp}(a)$ with $p_{i\rho}\, p_{j\nu}$

$$\gamma^\mu \not{p}_j \gamma^\rho \hat{P}_\pm \otimes \gamma_\mu \not{p}_i \gamma_\rho \hat{P}_\mp = 4 \not{p}_i \hat{P}_\pm \otimes \not{p}_j \hat{P}_\mp$$

Sandwich between $\bar{w}_1 [\ldots] w_2 \;\; \bar{w}_3 [\ldots] w_4$

$$\left[\gamma^\mu \not{p}_j \gamma^\rho \hat{P}_\pm\right]_{12} \left[\gamma_\mu \not{p}_i \gamma_\rho \hat{P}_\mp\right]_{34} = 4 \left[\not{p}_i \hat{P}_\pm\right]_{12} \left[\not{p}_j P_\mp\right]_{34}$$

Useful identities are obtained if $\not{p}$'s are anticommuted towards γ^μ (or γ_ρ)

① $p_j = p_1, \quad p_i = p_3$

$$\text{LHS} = [\gamma^\mu \not{p}_1 \gamma^\rho \hat{P}_\pm]_{12} [\gamma_\mu \not{p}_3 \gamma_\rho \hat{P}_\mp]_{34}$$

$$= [(-\not{p}_1\gamma^\mu + 2p_1^\mu)\gamma^\rho \hat{P}_\pm]_{12}[(-\not{p}_3 \gamma_\mu + 2p_{3\mu})\gamma_\rho \hat{P}_\mp]_{34}$$

$$= [(-\mathrm{sgn}_1 m_1 \gamma^\mu + 2p_1^\mu)\gamma^\rho \hat{P}_\pm]_{12}[(-\mathrm{sgn}_3 m_3\gamma_\mu + 2p_{3\mu})\gamma_\rho \hat{P}_\mp]_{34}$$

$$= -2i\,\mathrm{sgn}_1 m_1 [\sigma^{\rho\nu} p_{3\nu} \hat{P}_\pm]_{12}[\gamma_\rho P_\mp]_{34}$$

$$-2i\,\mathrm{sgn}_3 m_3 [\gamma_\rho \hat{P}_\pm]_{12}[\sigma^{\rho\nu} p_{1\nu}\hat{P}_\mp]_{34}$$

$$+4 p_1\cdot p_3[\gamma_\rho \hat{P}_\pm]_{12}[\gamma^\rho \hat{P}_\mp]_{34} \underset{RHS}{=} 4[\not{p}_3 \hat{P}_\pm]_{12}[\not{p}_1 \hat{P}_\mp]_{34}$$

② $p_j = p_2, \quad p_i = p_4$

$$\text{LHS} = [\gamma^\mu \not{p}_2 \gamma^\rho \hat{P}_\pm]_{12}[\gamma_\mu \not{p}_4 \gamma_\rho \hat{P}_\mp]_{34}$$

$$= [\gamma^\mu(-\gamma^\rho \hat{P}_\mp \not{p}_2 + 2p_2^\rho \hat{P}_\pm)]_{12}[\gamma_\mu(-\gamma_\rho \hat{P}_\pm \not{p}_4 + 2p_{4\rho}\hat{P}_\mp)]_{34}$$

$$= [\gamma^\mu(-\mathrm{sgn}_2 m_2 \gamma^\rho \hat{P}_\mp \not{p}_2 + 2p_2^\rho P_\pm)]_{12}[\gamma_\mu(-\mathrm{sgn}_4 m_4 \gamma_\rho \hat{P}_\pm + 2p_{4\rho}\hat{P}_\mp)]_{34}$$

$$= 2i\,\mathrm{sgn}\, m_2 [\sigma^{\mu\rho} p_{4\rho}\hat{P}_\mp]_{12}[\gamma_\mu \hat{P}_\mp]_{34}$$

$$+2i\,\mathrm{sgn}_4 m_4 [\gamma_\mu \hat{P}_\pm]_{12}[\sigma^{\mu\rho} p_{2\rho} P_\pm]_{34}$$

$$+4 p_2\cdot p_4 [\gamma_\mu \hat{P}_\pm]_{12}[\gamma^\mu \hat{P}_\mp]_{34} \underset{RHS}{=} 4[\not{p}_4 \hat{P}_\pm]_{12}[\not{p}_2 \hat{P}_\mp]_{34}$$

Summary

$$[\not p_4 \hat P_\pm]_{12} [\not p_1 \hat P_\pm]_{34} = p_1 \cdot p_4 [\gamma^\mu \hat P_\pm]_{12} [\gamma_\mu \hat P_\pm]_{34}$$
$$- \frac{1}{2} i \, \mathrm{sgn}\, n_1 \, m_1 [\sigma^{\mu\nu} p_{4\nu} \hat P_\pm]_{12} [\gamma_\mu \hat P_\pm]_{34}$$
$$+ \frac{1}{2} i \, \mathrm{sgn}\, n_4 \, m_2 [\gamma_\mu \hat P_\pm]_{12} [\sigma^{\mu\nu} p_{1\nu} \hat P_\mp]_{34} \qquad (1)$$

$$[\not p_3 \hat P_\pm]_{12} [\not p_2 \hat P_\pm]_{34} = p_2 \cdot p_3 \, [\gamma_\mu \hat P_\pm]_{12} [\gamma^\mu \hat P_\pm]_{34}$$
$$+ \frac{1}{2} i \, \mathrm{sgn}\, n_2 \, m_2 [\sigma^{\mu\nu} p_{3\nu} \hat P_\mp]_{12} [\gamma_\mu \hat P_\pm]_{34}$$
$$- \frac{1}{2} i \, \mathrm{sgn}\, n_3 \, m_3 [\gamma_\mu \hat P_\pm]_{12} [\sigma^{\mu\nu} p_{2\nu} \hat P_\pm]_{34} \qquad (2)$$

$$[\not p_3 \hat P_\pm]_{12} [\not p_1 \hat P_\mp]_{34} = p_1 \cdot p_3 \, [\gamma_\mu \hat P_\pm]_{12} [\gamma^\mu \hat P_\mp]_{34}$$
$$- \frac{1}{2} i \, \mathrm{sgn}\, n_1 \, m_1 [\sigma^{\mu\nu} p_{3\nu} \hat P_\pm]_{12} [\gamma_\mu \hat P_\mp]_{34}$$
$$- \frac{1}{2} i \, \mathrm{sgn}\, n_3 \, m_3 [\gamma_\mu \hat P_\pm]_{12} [\sigma^{\mu\nu} p_{1\nu} \hat P_\mp]_{34} \qquad (3)$$

$$[\not p_4 \hat P_\pm]_{12} [\not p_2 \hat P_\mp]_{34} = p_2 \cdot p_4 \, [\gamma_\mu \hat P_\pm]_{12} [\gamma^\mu \hat P_\mp]_{34}$$
$$+ \frac{1}{2} i \, \mathrm{sgn}\, n_2 \, m_2 [\sigma^{\mu\nu} p_{4\nu} \hat P_\mp]_{12} [\gamma_\mu \hat P_\mp]_{34}$$
$$+ \frac{1}{2} i \, \mathrm{sgn}\, n_2 \, m_4 [\gamma_\mu \hat P_\pm]_{12} [\sigma^{\mu\nu} p_{2\nu} \hat P_\pm]_{34} \qquad (4)$$

Pattern to observe:

Identity for same projector involves momenta of one u and one $\bar u$ spinor.
Identity for opposite projectors involves momenta of both u spinors or both $\bar u$ spinors.

To deal with structures like $[\not p_4 P_\pm]_{12} [\not p_1 P_\mp]_{34}$, must use energy-momentum conservation to replace e.g. $p_1 = p_2 + p_4 - p_3$.
Then identities can be applied.