Imposing Majorana conditions on u & v spinors: $C = i\gamma^2\gamma^0$

① $\psi^c = C\bar{\psi}^T$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_s \left[\hat{a}^\dagger_{\vec{p},s} \underbrace{C\bar{u}^T}_{v} e^{ip\cdot x} + \hat{a}_{\vec{p},s} \underbrace{C\bar{v}^T}_{u} e^{-ip\cdot x} \right]$$

$= \psi$. (Majorana condition)

$$\therefore \boxed{\begin{array}{lll} C\bar{u}^T = v & \text{and} & C\bar{v}^T = u \\ \text{or } u^c = v & & v^c = u \end{array}}$$

② $\overline{\psi^c} = -\psi^T C^{-1}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_s \left[\hat{a}_{\vec{p},s} \underbrace{(-u^T C^{-1})}_{\bar{v}} e^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p},s} \underbrace{(-v^T C^{-1})}_{\bar{u}} e^{-ip\cdot x} \right]$$

$= \bar{\psi}$ (Majorana condition)

$\therefore -u^T C^{-1} = \bar{v}$ and $v^T C^{-1} = \bar{u}$

or or

$$\boxed{\begin{array}{ll} u^T C^{-1} = -\bar{v} & v^T C^{-1} = \bar{u} \\ \overline{u^c} = \bar{v} & \overline{v^c} = u \end{array}}$$

These identities are valid for Dirac spinors. Only difference between Dirac and Majorana cases is the Fourier amplitude: $b, d \rightarrow a$.