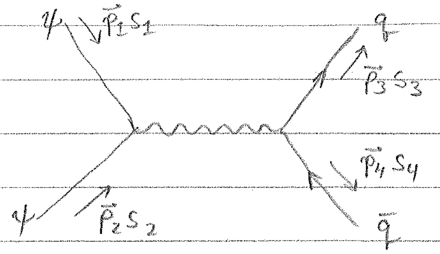
Example:

Process involving Majorana fermions: ψ

$$\psi\psi \to Z' \to \bar{q} q$$



Interaction lagrangian:

$$\mathcal{L}_I = +\tfrac{1}{2} g Z'_\mu \bar{\Psi} \gamma^\mu \gamma_5 \psi - g Z'_\mu \bar{q} \gamma^\mu q$$

Second-order matrix element

$$iT = \langle q_3 \bar{q}_4 | \, T e^{i\int d^4x \, \mathcal{L}_I} \, | \psi_1 \psi_2 \rangle$$

$$= \langle q_3 \bar{q}_4 | \, T \Big[1 + \dots + \underset{\text{Taylor}}{\frac{i^2}{2!}} \cdot \underset{\text{Binomial}}{2} \int d^4x \int d^4y \, \Big[\frac{g}{2} Z'_\mu \bar{\Psi} \gamma^\mu \gamma_5 \psi\Big]_{(x)} \Big[-g Z'_\nu \bar{q} \gamma^\nu q\Big]_{(y)} \, | \psi_1 \psi_2 \rangle$$

Can immediately contract Z' (as shown)

$$\overline{Z'_{(x)\mu} \, Z'_{(y)\nu}} = \int \frac{d^4 q}{(2\pi)^4} \, \frac{-i e^{-iq\cdot(x-y)}}{q^2 - m_{Z'}^2 + i\varepsilon} \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_{Z'}^2} \right)$$

Amplitude factorizes
because quark states and Majorana fermion states belong to different subspaces.

$$\text{Prefactor} = \frac{i^2}{2!} \cdot 2 \, \frac{-g^2}{2} (-i) = \frac{-ig^2}{2}$$

$$iT = \frac{-ig^2}{2} \int d^4x \, d^4y \int \frac{d^4 q}{(2\pi)^4} \, \frac{g_{\mu\nu} - q_\mu q_\nu / m_{Z'}^2}{q^2 - m_{Z'}^2 + i\varepsilon} \, \langle 0 | \bar{\Psi}_{(x)} \gamma^\mu \gamma_5 \psi_{(x)} | \psi_{\vec{p}_1} \psi_{\vec{p}_2} \rangle \langle q_{\vec{p}_3} \bar{q}_{\vec{p}_4} | \bar{q}_{(y)} \gamma^\nu q_{(y)} | 0 \rangle$$

T-ordering symbol irrelevant since
$\bar{\Psi} \gamma^\mu \gamma_5 \psi$ commutes with $\bar{q} \gamma^\nu q$.

Focus on $\langle 0 | \bar{\Psi}_{(x)} \gamma^\mu \gamma_5 \psi_{(x)} | \psi_{\vec{p}_1} \psi_{\vec{p}_2} \rangle$

Recall, plane-wave expansion for Majorana spinors:

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_s \left[\hat{a}_{\vec{k},s}\, u^{[s]}(\vec{k})\, e^{-ik\cdot x} + \hat{a}^\dagger_{\vec{k}}\, v^{[s]}(\vec{k})\, e^{ik\cdot x} \right]$$

$$\bar{\psi}(x) = \int \frac{d^3k}{(2\pi)^3} \;\;\text{"}\;\;\;\text{"}\;\; \left[\hat{a}^\dagger_{\vec{k},s}\, \bar{u}^{[s]}(\vec{k})\, e^{ik\cdot x} + a_{\vec{k}}\, \bar{v}^{[s]}(\vec{k})\, e^{-ik\cdot x} \right]$$

and one-particle states:

$$|\psi_{\vec{p}}\rangle = \sqrt{2\omega_{\vec{p}}}\,|0\rangle \qquad\qquad \{\hat{a}_{\vec{p},s}, \hat{a}^\dagger_{\vec{p}',s'}\} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')\,\delta_{ss'}$$

$\therefore$

$$\langle 0|\, \bar{\psi}_{(x)} \gamma^\mu \gamma_5 \psi_{(x)} \,|\psi_{\vec{p}_1} \psi_{\vec{p}_2}\rangle$$

$$= \sqrt{2\omega_{\vec{p}_1}}\sqrt{2\omega_{\vec{p}_2}}\, \langle 0|\, \bar{\psi}_{(x)} \gamma^\mu \gamma_5 \psi_{(x)}\, \hat{a}^\dagger_{\vec{p}_1 s_1} \hat{a}^\dagger_{\vec{p}_2 s_2} \,|0\rangle$$

Two contractions:

$$= \sqrt{2\omega_{\vec{p}_1}}\sqrt{2\omega_{\vec{p}_2}} \left[\langle 0|\, \bar{\psi} \gamma^\mu \gamma_5 \psi\, \hat{a}^\dagger_{\vec{p}_1 s_1} \hat{a}^\dagger_{\vec{p}_2 s_2} \,|0\rangle + \langle 0|\, \bar{\psi} \gamma^\mu \gamma_5 \psi\, \hat{a}^\dagger_{\vec{p}_1 s_1} \hat{a}^\dagger_{\vec{p}_2 s_2} \,|0\rangle \right]$$

To disentangle contractions.

$$= +\, \bar{v}^{[s_2]}(\vec{p}_2) \gamma^\mu \gamma_5 u^{[s_1]}(\vec{p}_1) \;-\; \bar{v}^{[s_1]}(\vec{p}_1) \gamma^\mu \gamma_5 u^{[s_2]}(\vec{p}_2)$$

Could proceed to square amplitude and compute traces

$\longrightarrow$ Would need atypical completeness relations:

$$\sum_s \bar{u}^{[s]}(\vec{p})\, \bar{v}^{[s]}(\vec{p}) = ?$$

$$\sum_s u^{[s]}(\vec{p})\, v^{[s]}(\vec{p}) = ?$$

Exercise

Instead, can manipulate second term to make it look like first term.

$$(2^{nd}\text{ term}) = \bar{v}^{[s_1]}(\vec{p}_1)\, \gamma^\mu \gamma_5\, u^{[s_2]}(\vec{p}_2)$$

Second term is a Dirac scalar — take transpose

$$(2^{nd}\ \text{term}) = \left[\bar{v}_1 \gamma^\mu \gamma_5 u_2\right]^T$$

γ_5 = symmetric.

$$= u_2^T \gamma_5 (\gamma^\mu)^T \bar{v}_1^T$$

$\boxed{\text{Apply: } u^T = -\bar{v} C \quad \text{and} \quad \bar{v}_1^T = C^{-1} u}$ $\qquad C = i\gamma^2\gamma^0$

$$= -\bar{v}_2 C \gamma_5 (\gamma^\mu)^T C^{-1} u_1$$

$$= -\bar{v}_2 \gamma_5 \underbrace{C(\gamma^\mu)^T C^{-1}}_{-\gamma^\mu} u_1$$

$$= +\bar{v}_2 \gamma_5 \gamma^\mu u_1$$

$$= -\bar{v}_2 \gamma^\mu \gamma_5 u_1 = -\text{First term.}$$

$\therefore$

$$\langle 0| \bar{\psi} \gamma^\mu \gamma_5 \psi | \psi_{p_1} \psi_{p_2} \rangle = 2\, \bar{v}_{s_2}(\vec{p}_2) \gamma^\mu \gamma_5 u_{s_1}(\vec{p}_1)$$