Free fermion propagator

$$\mathcal{L}_0 = \bar{\psi}\,(i\not{\partial} + C\, i\not{\partial}\,\gamma_5 - m_1 - i m_2 \gamma_5)\,\psi$$

$C, m_1, m_2 \in \mathbb{R}$
For $\mathcal{L}_0$ satisfying reality.

$\boxed{\text{Interpretation of } i\not{\partial}\gamma_5 :}$

leads to non-canonical normalization of left- & right- chiral fields:

Rewrite:

$$\mathcal{L}_0 = \bar{\psi}\left[i\not{\partial}(\hat{P}_L + \hat{P}_R) + C\, i\not{\partial}(-\hat{P}_L + \hat{P}_R) - m_1(\hat{P}_L + \hat{P}_R) - i m_2(-\hat{P}_L + \hat{P}_R)\right]\psi$$

$$\therefore\ \mathcal{L}_0 = \underbrace{(1-C)}_{\neq 1 \text{ in general}}\,\bar{\psi}_L\, i\not{\partial}\,\psi_L + \underbrace{(1+C)}_{\neq 1 \text{ in general}}\,\bar{\psi}_R\, i\not{\partial}\,\psi_R - (m_1 + i m_2)\,\bar{\psi}_L \psi_R - (m_1 - i m_2)\,\bar{\psi}_R \psi_L$$

Rescale ψ_L & ψ_R separately $\rightarrow$ <u>leads to parity-violating interactions.</u>

$$\psi_L \rightarrow \frac{1}{\sqrt{1-C}}\,\psi_L \qquad \psi_R \rightarrow \frac{1}{\sqrt{1+C}}\,\psi_R$$

$$\mathcal{L}_0 \longrightarrow \bar{\psi}_L\, i\not{\partial}\,\psi_L + \bar{\psi}_R\, i\not{\partial}\,\psi_R - \frac{m_1 + i m_2}{1+C}\,\bar{\psi}_L \psi_R - \frac{m_1 - i m_2}{1-C}\,\bar{\psi}_R \psi_L$$

$$= -\bar{\psi}\left(\frac{m_1 + i m_2}{1+C}\,\hat{P}_R + \frac{m_1 - i m_2}{1-C}\,\hat{P}_L\right)\psi$$

$$= -\bar{\psi}\left(\underbrace{\frac{1}{2}\frac{m_1 + i m_2}{1+C} + \frac{1}{2}\frac{m_1 - i m_2}{1-C}}_{\text{Real-valued} := m_R} + \underbrace{\left(\frac{1}{2}\frac{m_1 + i m_2}{1+C} - \frac{1}{2}\frac{m_1 - i m_2}{1-C}\right)}_{\text{Pure-imaginary} := i m_I}\gamma_5\right)\psi$$

$$\mathcal{L}_0 = \bar{\psi}\left(i\not{\partial} - m_R - i m_I \gamma_5\right)\psi$$

where $m_R = \dfrac{m_1 + C m_2}{1 - C^2} \in \mathbb{R}$ & $m_I = \dfrac{-C m_1 + m_2}{1 - C^2} \in \mathbb{R}$.

$\boxed{\text{Interpretation of } \; i m_I \gamma_5 :}$

Corresponds to chiral/complex mass.

Rewrite mass term:

$$\mathcal{L}_0 = \ldots - \underbrace{(m_R + i m_I)}_{\substack{\text{write in complex} \\ \text{polar form:} \\ = |m| e^{i\theta}}} \bar{\psi}_L \psi_R - \underbrace{(m_R - i m_I)}_{|m| e^{-i\theta}} \bar{\psi}_R \psi_L$$

where: $|m| = \sqrt{m_R^2 + m_I^2}$

$\theta = \tan^{-1}\left(\frac{m_I}{m_R}\right)$

inverse relations:

$m_R = |m| \cos\theta$, $m_I = |m| \sin\theta$

$$= \ldots - |m| e^{i\theta} \bar{\psi}_L \psi_R - |m| e^{-i\theta} \bar{\psi}_R \psi_L$$

Then, an axial rotation can remove phase: $\longrightarrow$ leads to CP-violating interactions.

$$\psi \xrightarrow{A} e^{\frac{i\alpha}{2}\gamma_5} \psi \qquad \bar{\psi} \rightarrow \bar{\psi} e^{\frac{i\alpha}{2}\gamma_5}$$

or equivalently,

$$\psi_L \longrightarrow e^{-i\alpha/2} \psi_L$$

$$\psi_R \longrightarrow e^{i\alpha/2} \psi_R$$

$$\mathcal{L}_0 \longrightarrow -|m| e^{i\theta + i\alpha} \bar{\psi}_L \psi_R - |m| e^{-i\theta - i\alpha} \bar{\psi}_R \psi_L$$

take $\theta = -\alpha$

$$= -|m| \bar{\psi}_L \psi_R - |m| \bar{\psi}_R \psi_L$$

$$= -|m| \bar{\psi} \psi.$$

Complex mass identity

$$e^{i\theta\gamma_5} = \sum_{n=0}^{\infty} \frac{(i\theta\gamma_5)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{(i\theta\gamma_5)^{2n}}{(2n)!} + \frac{(i\theta\gamma_5)^{2n+1}}{(2n+1)!} \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{(i\theta)^{2n}}{(2n)!} + \frac{(i\theta)^{2n+1}}{(2n+1)!}\gamma_5 \right]$$

$$= \cos\theta + (i\sin\theta)\gamma_5$$

Then complex mass term can be written:

$$\mathcal{L}_0 = \ldots - \bar\psi (m_R + i m_I \gamma_5)\psi \qquad \text{apply} \quad m_R = |m|\cos\theta, \quad m_I = |m|\sin\theta$$

$$= \ldots - \bar\psi \, |m| \underbrace{(\cos\theta + i\sin\theta\,\gamma_5)}_{\text{derived identity}}\psi \qquad (*)$$

$$= \ldots - \bar\psi \, |m| e^{i\theta\gamma_5}\psi$$

Then, an axial rotation $\psi \to e^{i\frac{\alpha}{2}\gamma_5}\psi$

$\bar\psi \to \bar\psi e^{\frac{i\alpha}{2}\gamma_5}$ (same sign in exp)

can remove θ if $\alpha = -\theta$.

NEGATIVE MASS:

If m_R is negative (and $m_I = 0$), this corresponds to $\theta = \pi$. (See $*$).

$$\mathcal{L}_0 = \ldots - \bar\psi(-|m_R|)\psi = \ldots - \bar\psi\,|m|\cos(\pi)\psi.$$

Therefore, an axial rotation with phase $\alpha = -\pi$ can render mass term positive.

Free propagator with complex mass:

$$\mathcal{L}_0 = \bar{\psi}\,\underbrace{(i\not\partial - m_R - i m_I \gamma_5)}_{\mathcal{O}}\,\psi$$

Propagator given by inverse of $\mathcal{O}$:

$$\tilde{\mathcal{O}} = \not p - m_R - i m_I \gamma_5$$

$$\tilde{S}^{-1} = -i\tilde{\mathcal{O}} = -i(\not p - m_R - i m_I \gamma_5)$$

schematically, $$\tilde{S}(\not p) = \frac{i}{\not p - m_R - i m_I \gamma_5} \times \underbrace{\left(\frac{\not p + m_R - i m_I \gamma_5}{\not p + m_R - i m_I \gamma_5}\right)}_{1}$$

$$= \frac{i(\not p + m_R - i m_I \gamma_5)}{p^2 \underbrace{- m_R^2 - m_I^2}_{-|m|^2} + i\epsilon} \Rightarrow \boxed{\text{pole at } p^2 = m_R^2 + m_I^2 - i\epsilon}$$

$$\boxed{\Rightarrow \tilde{S}(\not p) = \frac{i(\not p + m_R - i m_I \gamma_5)}{p^2 - |m|^2 + i\epsilon} = \frac{i(\not p + |m| e^{-i\theta\gamma_5})}{p^2 - |m|^2 + i\epsilon}} \qquad \theta = \tan^{-1}\left(\frac{m_I}{m_R}\right)$$

In position space,

$$S(x-y)_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not p + m_R - i m_I \gamma_5)_{\alpha\beta}}{p^2 - |m|^2 + i\epsilon} e^{-ip.(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(i\not\partial + m_R - i m_I \gamma_5)_{\alpha\beta}}{p^2 - |m|^2 + i\epsilon} e^{-ip.(x-y)}$$

$$= (i\not\partial + m_R - i m_I \gamma_5)_{\alpha\beta}\; G_F(|m|^2; x-y)$$