Poincaré group — isometry group of Minkowski space.

Def$^{\underline{n}}$:

$$x^\mu \longrightarrow x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{\text{active rotation}} x^\nu + \underset{\substack{\uparrow \\ \text{active translation.}}}{a^\mu} \qquad = \Lambda x + a \quad \text{(index free)}$$

composition law:

$$x \xrightarrow{1} \Lambda_1 x + a_1 \xrightarrow{2} \Lambda_2(\Lambda_1 x + a_1) + a_2$$

$$= \underbrace{\Lambda_2 \Lambda_1}_{\text{total rotation}} x + \underbrace{\Lambda_2 a_1 + a_2}_{\text{total translation.}}$$

composition

$$\boxed{\underbrace{(\Lambda_2, a_2)}_{\text{next group element}} \circ \underbrace{(\Lambda_1, a_1)}_{\text{first group element}} = \underbrace{(\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)}_{\text{composed element.}}}$$

Just as Lorentz group, Poincaré group decomposes into four disconnected parts, identified by $\det\Lambda$ and $\Lambda^0{}_0$.

Replace "L" with "P"

$P_+^\uparrow$ = proper orthochronus ⟵ contains identity element: $(\mathbb{1}_{4\times4}, 0)$

$P_+^\downarrow$ = proper nonorthochronus

~~$P_-^\uparrow$ = improper orthochronus~~

~~$P_-^\downarrow$ = improper nonorthochronus.~~

Inverse: $(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$

check: $(\Lambda, a) \circ (\Lambda^{-1}, -\Lambda^{-1}a) = (\Lambda\Lambda^{-1}, \Lambda(-\Lambda^{-1}a) + a) = (\mathbb{1}, 0)$

and

$(\Lambda^{-1}, -\Lambda^{-1}a) \circ (\Lambda, a) = (\Lambda^{-1}\Lambda, \Lambda^{-1}a - \Lambda^{-1}a) = (\mathbb{1}, 0)$

✓

To determine Poincaré algebra, consider a faithful representation of $P_+^\uparrow$

$$\underbrace{(\Lambda, a)}_{\text{element of } P_+^\uparrow} \xrightarrow[\text{"represented by"}]{} \underbrace{g(\Lambda, a)}_{\text{faithful representation}}$$

$$g(\Lambda_2, a_2)\, g(\Lambda_1, a_1) = g(\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2)$$
$$g^{-1}(\Lambda, a) = g(\Lambda^{-1}, -\Lambda^{-1} a)$$

definition of faithful representation.

$g(\Lambda, a): \underbrace{V}_{\text{vector space}} \longrightarrow V$ is bijective (one-to-one & onto) linear map.

Infinitesimally write, in neighborhood of identity:

$$g(\Lambda, a) = \mathbb{1}_V \underbrace{- \frac{i}{2}\omega_{\rho\sigma} M^{\rho\sigma}}_{\text{boosts/rotations}} + \underbrace{i a_\mu P^\mu}_{\text{translations}} \qquad (*)$$

Consider:

$$\underbrace{g^{-1}(\Lambda, 0)}_{\text{inverse}}\, g(\Lambda', a')\, g(\Lambda, 0) = g(\Lambda^{-1}, 0) \quad \underbrace{g(\Lambda', a')\, g(\Lambda, 0)}$$
$$= g(\Lambda^{-1}, 0) \quad g(\Lambda'\Lambda, a')$$
$$= g(\Lambda^{-1}\Lambda'\Lambda, \Lambda^{-1} a') \qquad (**)$$

Plug $(*)$ into $(**)$ keeping terms linear in ω & a.

$$\text{LHS} = g^{-1}(\Lambda, 0)\left[\mathbb{1} - \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} + i a_\mu P^\mu\right] g(\Lambda, 0)$$
$$= \mathbb{1} - \frac{i}{2}\omega'_{\mu\nu}\, g^{-1}(\Lambda, 0) M^{\mu\nu} g(\Lambda, 0) + i a'_\mu\, g^{-1}(\Lambda, 0) P^\mu g(\Lambda, 0)$$

$$\text{RHS} = \mathbb{1} - \frac{i}{2}(\Lambda^{-1}\omega'\Lambda)_{\rho\sigma} M^{\rho\sigma} + i(\Lambda^{-1}a')_\rho P^\rho$$
$$= \mathbb{1} - \frac{i}{2}(\Lambda^{-1})\,\omega'_{\mu\nu}\Lambda^\nu{}_\sigma M^{\rho\sigma} + i(\Lambda^{-1})_\rho{}^\mu a'_\mu P^\rho$$
$$= \mathbb{1} - \frac{i}{2}\omega'_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma M^{\rho\sigma} + i a'_\mu \Lambda^\mu{}_\rho P^\rho$$

Match terms:

$\omega'_{\mu\nu}$: $-\frac{i}{2}\, g^{-1}(\Lambda,0)\, M^{\mu\nu}\, g(\Lambda,0) = -\frac{i}{2}\,\Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma}\, M^{\rho\sigma}$

$$\boxed{g^{-1}(\Lambda,0)\, M^{\mu\nu}\, g(\Lambda,0) = \Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma}\, M^{\rho\sigma}}$$ $\Rightarrow$ $\hat{M}$ is antisymmetric (spin-1) tensor operator

EQN 1

α'_{μ} : $i\, g^{-1}(\Lambda,0)\, P^{\mu}\, g(\Lambda,0) = i\,\Lambda^{\mu}{}_{\rho}\, P^{\rho}$

$$\boxed{g^{-1}(\Lambda,0)\, P^{\mu}\, g(\Lambda,0) = \Lambda^{\mu}{}_{\rho}\, P^{\rho}}$$ $\Rightarrow$ $\hat{P}$ is a 4-vector

EQN 2

Explicitly, LHS of EQN 1 gives:

$$g^{-1}(\Lambda,0)\, M^{\mu\nu}\, g(\Lambda,0) = g(\Lambda^{-1},0)\, M^{\mu\nu}\, g(\Lambda,0)$$
$$= \left(\mathbb{1} + \tfrac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right) M^{\mu\nu} \left(\mathbb{1} - \tfrac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}\right)$$
$$= M^{\mu\nu} + \tfrac{i}{2}\omega_{\rho\sigma}\left[M^{\rho\sigma}, M^{\mu\nu}\right] + \mathcal{O}(\omega^2)$$
$$= M^{\mu\nu} - \tfrac{i}{2}\omega_{\rho\sigma}\left[M^{\mu\nu}, M^{\rho\sigma}\right]$$

and RHS gives

$$\Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma}\, M^{\rho\sigma} = \left(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}\right) M^{\rho\sigma}$$

inner: outer: lasts:

$$= M^{\mu\nu} + \omega^{\mu}{}_{\rho}\, M^{\rho\nu} + \omega^{\nu}{}_{\sigma}\, M^{\mu\sigma} + \mathcal{O}(\omega^2)$$
$$= M^{\mu\nu} + \omega_{\sigma\rho}\, g^{\sigma\mu}\, M^{\rho\nu} + \omega_{\rho\sigma}\, g^{\nu\rho}\, M^{\mu\sigma}$$
$$= M^{\mu\nu} - \omega_{\rho\sigma}\left(-g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}\right)$$

antisymmetrize $\rho \leftrightarrow \sigma$

$$= M^{\mu\nu} - \tfrac{1}{2}\omega_{\rho\sigma}\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}\right)$$

$$\Rightarrow \boxed{\left[M^{\mu\nu}, M^{\rho\sigma}\right] = -i\left(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho}\right)}$$ ✓

Lorentz algebra

Similarly LHS and RHS of EQN 2 gives:

$$g^{-1}(\Lambda, 0)\, P^\rho\, g(\Lambda, 0) = \Lambda^\rho{}_\nu P^\nu$$

$$g(\Lambda^{-1}, 0)\, P^\rho\, g(\Lambda, 0) = \Lambda^\rho{}_\nu P^\nu$$

$$\left(1 + \tfrac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right) P^\rho \left(1 - \tfrac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right) = \left(\delta^\rho{}_\nu + \omega^\rho{}_\nu\right) P^\nu$$

$$P^\mu + \underbrace{\tfrac{i}{2}\omega_{\mu\nu}\left[M^{\mu\nu}, P^\rho\right]} + O(\omega^2) = P^\rho + \omega^\rho{}_\nu P^\nu$$

$$= P^\mu + \omega_{\mu\nu}\, g^{\mu\rho} P^\nu$$

antisymmetrize $\mu \leftrightarrow \nu$

$$= P^\mu + \underbrace{\tfrac{1}{2}\omega_{\mu\nu}\left(g^{\mu\rho} P^\nu - g^{\nu\rho} P^\mu\right)}$$

MATCH

$$\Rightarrow \boxed{\left[M^{\mu\nu}, P^\rho\right] = -i\left(g^{\mu\rho} P^\nu - g^{\nu\rho} P^\mu\right)}$$

Thus, the Poincaré algebra is:

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right] = -i\left(g^{\mu\rho} M^{\nu\sigma} - g^{\mu\sigma} M^{\nu\rho} - g^{\nu\rho} M^{\mu\sigma} + g^{\nu\sigma} M^{\mu\rho}\right)$$

$$\left[M^{\mu\nu}, P^\rho\right] = -i\left(g^{\mu\rho} P^\nu - g^{\nu\rho} P^\mu\right)$$

$$\left[P^\mu, P^\nu\right] = 0 \qquad \text{(obvious: translations commute)}$$

Poincaré algebra in terms of $H, \vec{P}, \vec{J}, \vec{K}$

Lorentz algebra $\left\{ \begin{array}{l} [J_i, J_j] = i\epsilon_{ijk} J_k \\ [J_i, K_j] = i\epsilon_{ijk} K_k \\ [K_i, K_j] = -i\epsilon_{ijk} J_k \end{array} \right.$

$[J_i, J_j] = i\epsilon_{ijk} J_k$ Angular momenta $\equiv SO(3)$

$[J_i, K_j] = i\epsilon_{ijk} K_k$ $\Rightarrow$ Thomas precession (Boost is a 3-vector) (angular momentum transforms nontrivially)

$[K_i, K_j] = -i\epsilon_{ijk} J_k$ Minus sign due to metric $SO(3,1)$

$[J_i, H] = 0$ Angular momentum conserved

$[K_i, H] = -i P_i$ "Boostness" conserved: $\frac{dH}{dt} = \{K_i, H\} + \frac{\partial}{\partial t} K_i$

(because K_i contains explicit time dependence)

$[J_i, P_j] = -i\epsilon_{ijk} P_k$ Momentum is a 3-vector

$[K_i, P_j] = i\delta_{ij} H$

$[H, H] = 0$ Energy is conserved

$[P_i, H] = 0$ Momentum is conserved

$[P_i, P_j] = 0$ translations is abelian.