Casimir operators of the Poincaré algebra

Poincaré algebra:

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i(g^{\mu\rho}M^{\nu\sigma} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma} + g^{\nu\sigma}M^{\mu\rho})$$

$$[M^{\mu\nu}, P^{\rho}] = -i(g^{\mu\rho}P^{\nu} - g^{\nu\rho}P^{\mu})$$

$$[P^{\mu}, P^{\nu}] = 0$$

P^2 is a Casimir operator

Proof:

$$\begin{aligned}
[M^{\mu\nu}, P^2] &= [M^{\mu\nu}, P_\rho P^\rho] \\
&= [M^{\mu\nu}, P^\rho]P_\rho + P_\rho[M^{\mu\nu}, P^\rho] \\
&= -i(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu)P_\rho - iP_\rho(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu) \\
&= -i(\cancel{P^\nu P^\mu} - \cancel{P^\mu P^\nu}) - i(\cancel{P^\mu P^\nu} - \cancel{P^\nu P^\mu}) = 0 \quad \checkmark
\end{aligned}$$

$$\begin{aligned}
[P^\mu, P^2] &= [P^\mu, P^\nu P_\nu] \\
&= [P^\mu, P^\nu]P_\nu + P_\nu[P^\mu, P^\nu] = 0. \quad \checkmark
\end{aligned}$$

W^2 is a Casimir operator

Proof

$$\begin{aligned}
[M^{\mu\nu}, W^2] &= [M^{\mu\nu}, W_\rho W^\rho] \\
&= W_\rho[M^{\mu\nu}, W^\rho] + [M^{\mu\nu}, W_\rho]W^\rho \\
&= W_\rho(-i)(g^{\mu\rho}W^\nu - g^{\nu\rho}W^\mu) + (-i)(\delta^\mu{}_\rho W^\nu - \delta^\nu{}_\rho W^\mu)W^\rho \\
&= -i[\cancel{W^\mu W^\nu} - \cancel{W^\nu W^\mu} + \cancel{W^\nu W^\mu} - \cancel{W^\mu W^\nu}] = 0 \quad \checkmark
\end{aligned}$$

(of course since $W^2 \equiv$ L. scalar)

$$\begin{aligned}
[P^\mu, W^2] &= [P^\mu, W_\nu W^\nu] \\
&= \underbrace{[P^\mu, W_\nu]}_{0}W^\nu + W_\nu\underbrace{[P^\mu, W^\nu]}_{0} = 0 \quad \checkmark
\end{aligned}$$