Review of covariant formulation of electrodynamics

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} A^0$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Field strength (Faraday tensor)

$$F^{\mu}{}_{\nu} = \begin{pmatrix} 0 & + & + & + \\ + & 0 & + & - \\ + & - & 0 & + \\ + & + & - & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Dual

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \qquad \tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}$$

$$\epsilon^{0123} = +1 \qquad \vec{E}_i = F_{0i}$$

$$\epsilon_{0123} = -1 \qquad \vec{B}_i = -\epsilon_{ijk} F_{jk} \Rightarrow F_{ij} = -\epsilon_{ijk} \vec{B}_k$$

Lorentz-invariant combinations

$$F_{\mu\nu} F^{\mu\nu} = -2(\vec{E}^2 - \vec{B}^2)$$

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = -4 \vec{E}\cdot\vec{B}$$

$$= \partial_\mu \left[\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma}\right]$$

Bianchi Identity

$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$ (automatic by antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$)

Current

$$J^\mu = (\underset{\substack{\uparrow \\ \text{charge density}}}{\rho}, \underset{\substack{\uparrow \\ \text{current density}}}{\vec{J}})$$

For a single charged particle at position $\vec{X}(t)$ velocity $d\vec{X}(t)/dt$ and charge q.

$$\rho(t, x) = q\, \delta^{(3)}(\vec{x} - \vec{X}(t))$$

$$\vec{J}(t, \vec{x}) = q\, \delta^{(3)}(\vec{x} - \vec{X}(t)) \frac{d\vec{X}(t)}{dt}$$