Hamiltonian Mechanics — review

Lagrangian: $L = L(q^i, \dot{q}^i, \ldots)$

Conjugate momenta: $p_i = \dfrac{\partial L}{\partial \dot{q}^i}$

Hamiltonian: $H = p_i \dot{q}^i - L$ (elim. $\dot{q}_i$ in favor of p_i's)

Hamilton's equations of motion:

Vary the definition of the Hamiltonian,

$$\delta H = p_i \,\delta \dot{q}^i + \dot{q}^i \,\delta p_i - \frac{\partial L}{\partial q^i}\delta q^i - \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i - \frac{\partial L}{\partial t}\delta t$$

$$\approx \underbrace{\left(p_i - \frac{\partial L}{\partial \dot{q}^i}\right)}_{=0 \text{ (EOM)}} \delta \dot{q}^i - \dot{p}_i \,\delta q^i + \dot{q}^i \,\delta p_i - \frac{\partial L}{\partial t}\delta t \qquad 1$$

Vary the Hamiltonian as a general function: $H(q, p, t)$

$$\delta H = \frac{\partial H}{\partial q^i}\delta q^i + \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial t}\delta t \qquad 2$$

Equate 1 = 2

$$0 = \left(\frac{-\partial H}{\partial q^i} - \dot{p}_i\right)\delta q^i + \left(\frac{-\partial H}{\partial p_i} + \dot{q}^i\right)\delta p_i + \left(\frac{-\partial H}{\partial t} - \frac{\partial L}{\partial t}\right)\delta t$$

Since the variations are independent, we arrive at the following equations of motion:

$$\boxed{-\dot{p}_i = \frac{\partial H}{\partial q^i} \quad \text{and} \quad \dot{q}^i = \frac{\partial H}{\partial p_i}} \qquad \frac{\partial H}{\partial t} = \frac{-\partial L}{\partial t}$$

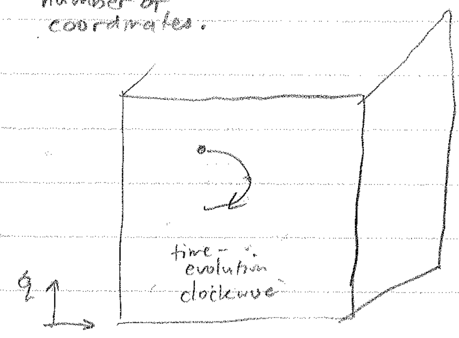
$$\frac{\partial H}{\partial q_i} + \dot{p}_i = 0 \qquad \dot{q}_i - \frac{\partial H}{\partial p_i} = 0$$

$$S = \int_{t_1}^{t_2} dt\, L(q_i, \dot{q}_i) \qquad i = 1, \ldots, N$$

number of coordinates.

Euler-Lagrange:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$



configuration space.

If $L = L(q(t), \dot{q}(t))$, (and indep of t)

$$\frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j}\dot{q}_i - \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_i = 0$$

or

$$\ddot{q}_i \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j}\dot{q}_i$$

Inertial mass matrix.

Accelerations are uniquely determined by positions and velocities at a given time, t iff the matrix $\partial^2 L / \partial \dot{q}_i \partial \dot{q}_j$ can be inverted to yield:

$$\ddot{q}_i = \frac{\partial L}{\partial q_i}\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)^{-1} - \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j}\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right)^{-1}\dot{q}_i$$

This is important since forces should be functions of positions & velocities that are known at each time, t. This is what yields predictable dynamics.

On the other hand, if inertial mass matrix $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ is not invertible, then accelerations will not be known uniquely.

over

TO HAMILTONIAN

Perform a Legendre transformation: Define momenta: $p_i = \frac{\partial L}{\partial \dot{q}_i}$

However a passage over to the Hamiltonian requires us to eliminate $\dot{q}_i$ in favor of p_i. However, in order for this to work, $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ needs to be invertible.

To HAMILTONIAN

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ A & B \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} \qquad \text{ie. } p_i = A_{ij} q_j + B_{ij} \dot{q}_j$$

Where for normal Lagrangians, quadratic in velocities,

$$\mathbb{A} \equiv \mathbb{A}(q) \qquad \text{and} \qquad \mathbb{B} \equiv \mathbb{B}(q)$$

To eliminate velocities in favor of momenta, need inverse mapping:

$$\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ -\mathbb{B}^{-1} \mathbb{A} & \mathbb{B}^{-1} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

the inverse mapping exists if $\mathbb{B}^{-1}$ exists.

Since $\vec{p}_i = \mathbb{A}(q)_{ij} q_j + \mathbb{B}(q)_{ij} \dot{q}_j$

$$\frac{\partial p_i}{\partial \dot{q}_j} = \mathbb{B}(q)_{ij}$$

but $\vec{p}_i = \dfrac{\partial L}{\partial \dot{q}_i}$

so $$\boxed{\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \mathbb{B}(q)_{ij}}$$

$\Rightarrow \dfrac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ needs to be invertible.

Action in Canonical form

$$H = \dot{x}p - L$$

or

$$L = \dot{x}p - H$$

If Ω^{ab} is invertible, I can write the action in Hamilton's form:

$$S = \int dt \left(\dot{x}p - H(x,p) \right)$$

$$= \int dt \left(\underbrace{\tfrac{1}{2}(p\dot{x} - x\dot{p})}_{\text{symplectic term}} - H(x,p) \right)$$

The Euler-Lagrange equations of motion derived from this action are Hamilton's equations of motion.

① $$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

$$-\frac{1}{2}\dot{p} - \frac{\partial H}{\partial x} - \frac{d}{dt}\left(\frac{1}{2}p\right) = 0$$

$$\Rightarrow \quad \dot{p} = -\frac{\partial H}{\partial x}$$

② $$\frac{\partial L}{\partial p} - \frac{d}{dt}\frac{\partial L}{\partial \dot{p}} = 0$$

$$\frac{1}{2}\dot{x} - \frac{\partial H}{\partial p} - \frac{d}{dt}\left(-\frac{1}{2}x\right) = 0$$

$$\dot{x} = \frac{\partial H}{\partial p}.$$

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