Maxwell least action principle

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu$$

Euler-Lagrange equation of motion:

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$$

rewrite: $\mathcal{L} = -\frac{1}{2} \partial_\mu A_\nu F^{\mu\nu} - A_\mu J^\mu$

$$= -\frac{1}{2} g^{\mu\rho} g^{\nu\sigma} \partial_\mu A_\nu (\partial_\rho A_\sigma - \partial_\sigma A_\rho) - g^{\mu\nu} A_\mu J_\nu$$

Then: $\frac{\partial \mathcal{L}}{\partial A_\nu} = -J^\nu$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \partial_\mu \left(-F^{\mu\nu} \right)$$

So equation of motion is:

$$-J^\nu + \partial_\mu F^{\mu\nu} = 0$$

or

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu}$$

Equation of motion $\underline{\partial_\mu F^{\mu\nu} = J^\nu}$ in component form:

Use: $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ $\quad J^\nu = (\rho, J_x, J_y, J_z)$

$F^{ij} = -\epsilon^{ijk} B^k$

$F^{0i} = -E^i$ and $F^{i0} = E^i$

$\underline{\nu = 0:}$

$\partial_\mu F^{\mu 0} = J^0$

or $\partial_i F^{i0} = J^0$

$\partial_i (E^i) = \rho$

$\boxed{\vec{\nabla} \cdot \vec{E} = \rho}$

Gauss' Law

$\underline{\nu = j = \{1, 2, 3\}}$

$\partial_\mu F^{\mu j} = J^j$

$\partial_0 F^{0j} + \partial_i F^{ij} = J^j$

$\partial_0 (-E^j) + \partial_i (-\epsilon^{ijk} B^k) = J^j$

$-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = \vec{J}$

$\boxed{\vec{\nabla} \times \vec{B} = \vec{J} + \frac{\partial \vec{E}}{\partial t}}$

Ampere's law
with Maxwell's correction.

$\underline{\partial_\mu \tilde{F}^{\mu\nu} = 0}$ in component form:

Use $\tilde{F}^{ij} = \epsilon^{ijk} E^k$

$\tilde{F}^{0i} = -B^i$ and $\tilde{F}^{i0} = B^i$

$\underline{\nu = 0:}$

$\partial_\mu \tilde{F}^{\mu 0} = 0$

$\partial_i \tilde{F}^{i0} = 0$

$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$

No magnetic
monopoles

$\nu = j = \{1, 2, 3\}$

$\partial_\mu \tilde{F}^{\mu j} = 0$

$\partial_0 \tilde{F}^{0j} + \partial_i \tilde{F}^{ij} = 0$

$\partial_0 (-B^j) + \partial_i (\epsilon^{ijk} E^k) = 0$

$-\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{E} = 0$

$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

Faraday's law of
induction.