Gauge Fixing

$$Z = \int \mathcal{D}A\, e^{iS[A]} = \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A\, e^{iS[A]} \overset{①}{\det\left(\frac{\delta G^a[A'(x)]}{\delta \alpha^b(y)}\right)} \overset{②}{\delta(G^a[A])}$$

Lorenz Gauges: Does not specify a unique A_μ. Can still make a "restricted" gauge transformation.

→ choose generalized Lorenz gauge condition:

$$G^a[A(x)] = \underbrace{\partial^\mu A^a_\mu(x)}_{\mathcal{F}} - \omega^a(x)$$

← Arbitrary "classical" function dependent on x.

$\mathcal{F} \equiv$ Fixing condition (vanishes for physical states)

Then

$$G^a[A'(x)] = \partial^\mu\left(A^a_\mu - \frac{1}{g}\partial_\mu \alpha^a - f^{abc}\alpha^b A^c_\mu\right) - \omega^a(x)$$

$$= \partial^\mu A^a_\mu - \frac{1}{g}\partial^\mu\partial_\mu \alpha^a - f^{abc}(\partial^\mu \alpha^b) A^c_\mu - f^{abc}\alpha^b \partial^\mu A^c_\mu - \omega^a(x)$$

At this point, we are done fixing the gauge. However, in order to perform calculations, the δ-functional & determinant need to be lifted into the exponent.

① $$\frac{\delta G^a[A'(x)]}{\delta \alpha^b(y)} = \frac{\partial G}{\partial \alpha^b}\delta(x-y) + \frac{\partial G}{\partial(\partial^\mu \alpha^b)}\partial^\mu\left(\delta(x-y)\,\cdot\right) + \frac{\partial G}{\partial(\partial_\mu\partial_\nu \alpha^b)}\partial_\mu\partial_\nu(\delta(x-y)\,\cdot$$

$$= -f^{abc}\partial^\mu A^c_\mu\,\delta(x-y) - f^{abc}A^c_\mu \underset{\text{hungry derivative}}{\partial^\mu}(\delta(x-y)\,\cdot) - \frac{1}{g}\delta^{ab} g^{\mu\nu} \underset{\text{hungry derivatives}}{\partial_\mu\partial_\nu}(\delta(x-y)\,\cdot)$$

$$= -f^{abc}\partial^\mu\left(A^c_\mu\,\delta(x-y)\,\cdot\right) - \frac{1}{g}\delta^{ab}\partial^\mu\partial_\mu\left(\delta(x-y)\,\cdot\right)$$

$$= -\frac{1}{g}\partial^\mu\left[\delta^{ab}\partial_\mu + g f^{abc}A^c_\mu\right]\delta(x-y)\,\cdot$$

$$\equiv -\frac{1}{g}\partial^\mu D^{ab}_\mu\,\delta(x-y)\,\cdot$$

Then $$\det\left(\frac{\delta G^a[A'(x)]}{\delta\alpha^b(y)}\right) = \det\left(-\frac{1}{g}\partial^\mu D^{ab}_\mu\,\delta(x-y)\,\cdot\right)$$

$$= \int \mathcal{D}\eta\,\mathcal{D}\bar{\eta}\; e^{i\int d^4x\, d^4y\left[\bar{\eta}^a(x)\,\underbrace{\tfrac{-1}{g}}_{\text{absorb into normalization for } \bar{\eta}}\partial^\mu D^{ab}_\mu\left(\delta(x-y)\,\eta^b(y)\right)\right]}$$

$$= \int \mathcal{D}\eta\,\mathcal{D}\bar{\eta}\; e^{i\int d^4x\, d^4y\left[-\bar{\eta}^a(x)\,\partial^\mu D^{ab}_\mu\left(\delta(x-y)\,\eta^b(y)\right)\right]}$$

η & $\bar{\eta}$ $\equiv$ (independent) Grassman valued fields (Faddeev-Popov ghosts), satisfying reality* condition

*Not in BRST formalism: $\bar{\eta}$ is "imaginary"

Simplify exponent:

$$= i\int d^4x \int d^4y \, -\bar{\eta}^a(x) \; \partial^\mu D_\mu^{ab}\left(\delta(x-y)\,\eta^b(y)\right)$$

$$= i\int d^4x \int d^4y \, -\bar{\eta}^a(x) \; \partial^\mu\left(\partial_\mu \delta^{ab} + g f^{abc} A_\mu^c\right)\delta(x-y)\,\eta^b(y)$$

$$= i\int d^4x \int d^4y \left[-\bar{\eta}^a(x)\,\delta^{ab}\,\underbrace{\partial^\mu\partial_\mu}\left(\delta(x-y)\,\eta^b(y)\right) - \bar{\eta}^a(x)\, g f^{abc}\,\partial^\mu\left(A_\mu^c(x)\,\delta(x-y)\,\eta^b(y)\right)\right]$$

- Integrate by parts twice, exposing $\delta(x-y)$
- Integrate over y, fixing $y \to x$
- Integrate by parts once, putting ∂ on η.
  $\Rightarrow$ (gained overall -1)

- Integrate by parts once
- Integrate over y fixing $y \to x$
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$$= i\int d^4x \left[\partial_\mu \bar{\eta}^a \, \partial^\mu \eta^a + g f^{abc}\left(\partial^\mu \bar{\eta}^a\right)\eta^b A_\mu^c\right]$$

② Now work on δ-functional:

$$\delta(G[A]) = \delta\left(\partial^\mu A_\mu^a(x) - \omega^a(x)\right)$$

The functional integral, Z, does not depend on the form of $\omega^a(x)$ (how could it? It was introduced by multiplying by $1 = \int \mathcal{D}\alpha \, \Delta_G \, \delta(G(A))$). I can freely divide the integral into two halves — each involving a different form for $\omega^a(x)$:

$$Z = \frac{1}{2}\int \mathcal{D}\alpha \int \mathcal{D}A \; e^{iS[A]} \det\left(\frac{\delta G^a[A'(x)]}{\delta\alpha(y)}\right) \delta\left(\partial^\mu A_\mu^a - \omega_1^a(x)\right)$$

$$+ \frac{1}{2}\int \mathcal{D}\alpha \int \mathcal{D}A \; e^{iS[A]} \det\left(\frac{\delta G^a[A'(x)]}{\delta\alpha(y)}\right) \delta\left(\partial^\mu A_\mu^a - \omega_2^a(x)\right).$$

Similarly, I can "continuously" divide up the functional integral, with each "term" involving an "infinitesimally" different form for $\omega^a(x)$, and then "continuously" add them back together as long as the overall normalization factor is unchanged.

$$Z = \int \mathcal{D}\omega^a \; \underbrace{e^{-i\int d^4x \frac{(\omega^a(x))^2}{2\xi}}}_{\text{Gaussian normalization}} \int \mathcal{D}\alpha \int \mathcal{D}A \; e^{iS[A]} \det\left(\frac{\delta G^a[A'(x)]}{\delta\alpha^b(y)}\right) \delta\left(\partial^\mu A_\mu^a - \omega^a(x)\right)$$

Explores the space of all $\omega^a(x)$.

Integrate over $\omega^a(x)$, using δ-functional, fixing $\omega^a(x) \longrightarrow \partial^\mu A^a_\mu(x)$

$$Z = \int \mathcal{D}\alpha \int \mathcal{D}A \; e^{iS[A]} \; e^{-i\int d^4x \frac{1}{2\xi}(\partial . A)^2} \det\left(\frac{\delta G^a[A'(x)]}{\delta \alpha^b(y)}\right)$$

$$= \underbrace{\int \mathcal{D}\alpha}_{\substack{\shortparallel \\ \left(\text{volume of gauge group}\right)\left(\text{volume of space}\right)}} \int \mathcal{D}A \, \mathcal{D}\eta \, \mathcal{D}\bar{\eta} \; e^{iS[A] - i\int d^4x \frac{1}{2\xi}(\partial . A)^2 + \partial_\mu \bar{\eta}\, \partial^\mu \eta + g f^{abc}(\partial^\mu \bar{\eta}^a)\eta^b A^c_\mu}$$

$\longleftarrow$ This infinite factor gets divided out. $\langle 0 | .. | 0 \rangle = \left.\dfrac{\frac{\partial}{\partial J} \cdot Z[J]}{Z[0]}\right|_{J=0}$

The resulting gauge-fixed Lagrangian is:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} - \frac{1}{2\xi}(\partial . A)^2 + \partial_\mu \bar{\eta}\, \partial^\mu \eta + g f^{abc}(\partial^\mu \bar{\eta}^a)\eta^b A^c_\mu$$

$$\boxed{\begin{aligned} \mathcal{L} \equiv\; & \frac{1}{2} A^a_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A^a_\nu + \frac{1}{2\xi} A^a_\mu \partial^\mu \partial^\nu A^a_\nu + \partial_\mu \bar{\eta}^a \partial^\mu \eta^a \\ & + g f^{abc} \partial_\mu A^a_\nu A^{\mu b} A^{\nu c} - \frac{g^2}{4} f^{abc} f^{cde} A^a_\mu A^b_\nu A^{\mu d} A^{\nu e} + g f^{abc} (\partial^\mu \bar{\eta}^a) \eta^b A^c_\mu . \end{aligned}}$$

For a general gauge fixing condition: $G^a[A] = \mathcal{F}^a(x) - \omega^a(x)$,
we have

Arbitrary gauge fixing condition.

$$Z = \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A \; e^{iS[A]} \det\left(\frac{\delta \mathcal{F}'^a(x)}{\delta \alpha^b(y)}\right) \delta\left(\mathcal{F}^a(x) - \omega^a(x)\right)$$

$$\vdots$$

$$\Rightarrow \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu\, a} - \frac{1}{2\xi}(\mathcal{F}^a)^2 + \int d^4y \; \bar{\eta}^a(x) \left[\frac{\delta \mathcal{F}'^a(x)}{\delta \alpha^c(y)}\right] \eta^c(y)$$

differential operators in
here are "hungry" derivatives
— act on everything to the right.