

(Lt Hoff)

Gauge Fixing in a Spontaneously Broken Non-abelian Gauge Theory -  $R_{\xi}$  gauge

Working in Yang-Mills theory (gauge group,  $G$ , consisting of multiple simple group factors), with scalars in various representations of  $G$ .

This leads to factor of 2 of Casimir quadratics.

- Convenient to put all scalars in the real representation,  $\Phi_i(x)$   
 $\Rightarrow$  Then all generators,  $T_{ij}^a$ , are purely imaginary and antisymmetric in  $ij$ .

Gauge Transformation rule for  $\Phi_i(x)$  (important for FP det.)

$$\Phi_i(x) \rightarrow (\mathbb{1}_{ij} + i\alpha^a(x)T_{ij}^a) \Phi_j(x)$$

Purely imaginary generators

$\alpha^a(x) \equiv$  parameters of transformation.

- factor out a  $-i$ , to get real generators:  $T^a = -i T_{Re}^a$

Then,  $\Phi_i(x) \rightarrow (\mathbb{1} + \alpha^a(x)T_{Re}^a)_{ij} \Phi_j(x)$        $[T_{Re}^a, T_{Re}^b] = -f^{abc}T_{Re}^c$       Purely real (still antisymm.)

Then covariant derivatives become:

$$(D_{\mu})_{ij} \Phi_j = (\delta_{ij} \partial_{\mu} + iA_{\mu}^a (gT^a)_{ij}) \Phi_j = (\delta_{ij} \partial_{\mu} + A_{\mu}^a (gT_{Re}^a)_{ij}) \Phi_j$$

And gauge fields transform like:

$$A_{\mu}^a \rightarrow A_{\mu}^a - \partial_{\mu} \frac{\alpha^a}{g} - f^{abc} \alpha^b A_{\mu}^c \equiv A_{\mu}^a - \left( D_{\mu}^{ab} \frac{\alpha^b}{g} \right), \text{ as usual.}$$

n.b.  $A_{\mu}^a \rightarrow A_{\mu}^a \oplus \partial_{\mu} \frac{\alpha^a}{g} \oplus f^{abc} \alpha^b A_{\mu}^c \equiv A_{\mu}^a + D_{\mu}^{ab} \frac{\alpha^b}{g}$   
 is used if  $D_{\mu} = \partial_{\mu} \ominus iA_{\mu}^a (gT^a)_{ij}$        $\uparrow$  unchanged

$\nwarrow$  gauge covariant derivative in the adjoint rep:  
 $D_{\mu}^{ab} = \partial_{\mu} \delta^{ab} + g f^{abc} A_{\mu}^c$

Lagrangian:

$$\mathcal{L} = \frac{1}{2} \underbrace{(D_{\mu} \Phi)}_{\text{expand}} (D^{\mu} \Phi) - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - V(\Phi)$$

$$= \frac{1}{2} (\partial_{\mu} \Phi_i + A_{\mu}^a g T_{ij}^a \Phi_j) (\partial^{\mu} \Phi_i + A^{\mu b} g T_{ik}^b \Phi_k) - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - V(\Phi)$$

Now expand this.

Note:  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c$  for non-abelian gauge theories.

Then:

$$\begin{aligned}
 -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} &= -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) (\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) + \frac{1}{4} \times 2g f^{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A^{\mu b} A^{\nu c} \\
 &\quad - \frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} \\
 &= -\frac{1}{2} \partial_\mu A_\nu^a (\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) + \underbrace{g f^{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c}}_{\text{"}gA^3\text{"}} - \underbrace{\frac{g^2}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e}}_{\text{"}g^2 A^4\text{"}} \\
 &\quad \leftarrow \text{Integrate by parts} \\
 &= +\frac{1}{2} A_\nu^a (\partial_\mu \partial^\mu A^{\nu a} - \partial_\mu \partial^\nu A^{\mu a}) + \dots \\
 &= \frac{1}{2} A_\mu^a (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu^a + \text{"}gA^3\text{"} + \text{"}g^2 A^4\text{"}.
 \end{aligned}$$

Substitute back to obtain the Lagrangian:

$$\begin{aligned}
 \mathcal{L} &= \frac{1}{2} \partial_\mu \Phi_i \partial^\mu \Phi_i + A^{\mu b} \partial_\mu \Phi_i g T_{ij}^b \Phi_j + \frac{1}{2} A_\mu^a A^{\mu b} (g T^a \Phi)_i (g T^b \Phi)_i - V(\Phi) \\
 &\quad + \frac{1}{2} A_\mu^a (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu^a + gA^3 + g^2 A^4.
 \end{aligned}$$

Suppose  $\Phi$  spontaneously acquires a nonzero vacuum expectation value in some of its components.

Write  $\Phi_i(x) = \langle \phi \rangle_i + \phi_i(x)$   $\begin{cases} \delta \phi_i(x) = \alpha^a T_{ij}^a (\langle \phi \rangle + \phi)_j \\ \delta \langle \phi \rangle_i = 0 \end{cases}$

Divide\* the space of values for  $\phi_i(x)$  into two subspaces:

- ① The space spanned by (non-zero) directions  $T^a \langle \phi \rangle_i$   $\leftarrow$  a such vectors;  $i$  are these vectors' components.
  - field fluctuations in these directions are eaten Goldstone bosons.
- ② Space spanned by directions orthogonal to the  $T^a \langle \phi \rangle_i$ . (ie  $T^a \langle \phi \rangle_i$  vanishes)
  - field fluctuations in these directions are the physical uneaten scalars.

\* [this division cannot be made if  $\langle \phi \rangle$  doesn't minimize  $V(\Phi)$ ]

• Shift:  $\Phi \rightarrow \langle \phi \rangle + \phi$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i + A^{\mu b} \partial_\mu \phi_i \overline{gT}_{ij}^b (\langle \phi \rangle + \phi)_j + \frac{1}{2} A_\mu^a A^{\mu b} (\overline{gT}^a (\langle \phi \rangle + \phi))_i (\overline{gT}^b (\langle \phi \rangle + \phi))_i - V(\langle \phi \rangle_i + \phi_i) + \frac{1}{2} A_\mu^a (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu^a + g A^3 + g^2 A^4$$

Collect terms quadratic in fields (higher order in fields give rise to interactions, and are not relevant for now).

$$\mathcal{L}^{(Quad)} = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i + \overbrace{A^{\mu a} \partial_\mu \phi_i [\overline{gT}^a \langle \phi \rangle]_i}^{\partial_\mu A^\mu - \phi \text{ mixing term}} + \frac{1}{2} \underbrace{[\overline{gT}^a \langle \phi \rangle]_i [\overline{gT}^b \langle \phi \rangle]_i}_{\text{Mass matrix for gauge fields}} A_\mu^a A_\nu^b - \frac{1}{2} \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} \Big|_{\Phi = \langle \phi \rangle} \phi_i \phi_j + \frac{1}{2} A_\mu^a (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu^a$$

The second term,  $A^{\mu a} \partial_\mu \phi_i [\overline{gT}^a \langle \phi \rangle]_i$ , mixes gauge bosons with goldstone bosons (for nonzero  $\overline{gT}^a \langle \phi \rangle$ ).

• When quantizing the system,  $\mathcal{L}_{GF} = -\frac{1}{2\xi} (F^a)^2$  is added.

Choose  $F^a = (\partial^\mu A_\mu^a - \xi [\overline{gT}^a \langle \phi \rangle]_i \phi_i)$ , ← 4 Hooft gauge ( $R_\xi$ )

$$\begin{aligned} \mathcal{L}_{GF} &= -\frac{1}{2\xi} (\partial^\mu A_\mu^a)^2 - \frac{\xi}{2} [\overline{gT}^a \langle \phi \rangle]_i \phi_i [\overline{gT}^a \langle \phi \rangle]_j \phi_j + \partial_\mu A^{\mu a} [\overline{gT}^a \langle \phi \rangle]_i \phi_i \\ &\quad - \frac{1}{2\xi} \partial^\mu A_\mu^a \partial^\nu A_\nu^a \\ &= + \frac{1}{2\xi} A_\mu^a \partial^\mu \partial^\nu A_\nu^a - \frac{\xi}{2} [\overline{gT}^a \langle \phi \rangle]_i \phi_i [\overline{gT}^a \langle \phi \rangle]_j \phi_j - A^{\mu a} [\overline{gT}^a \langle \phi \rangle]_i \partial_\mu \phi_i \end{aligned}$$

Integrate by parts (twice)

Combines with gauge boson kinetic term altering vector boson propagator

Gives  $\xi$ -dep. masses to goldstone bosons

Cancels  $\partial_\mu A^\mu - \phi$  mixing term.

• Now, work on the compensating Fadeev-Popov determinant: need gauge transformed fixing condition.

$$\mathcal{F}^a(A', \phi') = \partial^\mu A_\mu'^a - \xi [\overline{gT}^a \langle \phi \rangle]_i \phi'_i$$

gauge transformed

These fields are gauge transformed with parameter  $\alpha(x)$

$\frac{\delta \mathcal{F}^a}{\delta \alpha}$

$$\begin{aligned}
 F^a(A', \phi') &= \partial^\mu (A_\mu - \overset{\partial_\mu + g f^{abc} \alpha^b A_\mu^c}{D_\mu} \frac{\alpha^a}{g}) - \xi [g T^a \langle \phi \rangle]_i (\phi_i + \alpha^b T_{ij}^b (\langle \phi \rangle + \phi)_j) \\
 &= \partial^\mu A_\mu^a - \partial^\mu \partial_\mu \frac{\alpha^a}{g} - f^{abc} (\partial^\mu \alpha^b) A_\mu^c - f^{abc} \alpha^b (\partial^\mu A_\mu^c) \\
 &\quad - \xi (g T^a \langle \phi \rangle)_i \phi_i - \xi (g T^a \langle \phi \rangle)_i (\alpha^b T_{ij}^b (\langle \phi \rangle + \phi)_j)
 \end{aligned}$$

Now use  $\frac{\delta \mathcal{F}^a(x)}{\delta \alpha^b(y)} = \frac{\partial \mathcal{F}^a}{\partial \alpha^b} \delta^{(4)}(x-y) + \frac{\partial \mathcal{F}^a}{\partial (\partial^\mu \alpha^b)} \partial^\mu \delta^{(4)}(x-y) + \frac{\partial \mathcal{F}^a}{\partial (\partial^\mu \partial^\nu \alpha^b)} \partial^\mu \partial^\nu \delta^{(4)}(x-y)$

So,

$$\begin{aligned}
 \frac{\delta \mathcal{F}^a(x)}{\delta \alpha^b(y)} &= -f^{abc} (\partial^\mu A_\mu^c) \delta^{(4)}(x-y) - \xi (g T^a \langle \phi \rangle)_i T_{ij}^b (\langle \phi \rangle + \phi)_j \delta^{(4)}(x-y) \\
 &\quad - f^{abc} A_\mu^c \partial^\mu \delta^{(4)}(x-y) - \frac{\delta^{ab}}{g} g_{\mu\nu} \partial^\mu \partial^\nu \delta^{(4)}(x-y) \\
 &= \left[ -\frac{\delta^{ab}}{g} \partial^2 - f^{abc} \partial^\mu A_\mu^c - f^{abc} A_\mu^c \partial^\mu \leftarrow = \left( -\partial^\mu \frac{D_\mu^{ab}}{g} \right) \right. \\
 &\quad \left. - \xi (g T^a \langle \phi \rangle)_i T_{ij}^b (\langle \phi \rangle + \phi)_j \right] \delta^{(4)}(x-y)
 \end{aligned}$$

Then,  $\det \left( \frac{\delta \mathcal{F}^a(x)}{\delta \alpha^b(y)} \right) \rightarrow \int \mathcal{D}\bar{\eta} \mathcal{D}\eta e^{i \int d^4x \left[ -\bar{\eta}^a \left( \partial^\mu \frac{D_\mu^{ab}}{g} + \xi (g T^a \langle \phi \rangle) T (\langle \phi \rangle + \phi) \right) \eta^b \right]}$   
 (missing steps shown in "Gauge Fixing" notes)

Hence, the ghost Lagrangian is:

$$\begin{aligned}
 \mathcal{L}_{\text{ghost}} &= -\bar{\eta}^a \left[ \frac{\delta^{ab}}{g} \partial^2 + f^{abc} \partial^\mu (A_\mu^c \cdot) + \xi (g T^a \langle \phi \rangle)_i T_{ij}^b (\langle \phi \rangle + \phi)_j \right] \eta^b \\
 &= -\frac{1}{g} \bar{\eta}^a \left[ \delta^{ab} \partial^2 + g f^{abc} \partial^\mu (A_\mu^c \cdot) + \xi (g T^a \langle \phi \rangle)_i g T_{ij}^b (\langle \phi \rangle + \phi)_j \right] \eta^b \\
 &\quad \text{absorb only } 1/2
 \end{aligned}$$

Now integrate by parts in first two terms:

$$\begin{aligned}
 &= \partial_\mu \bar{\eta}^a \partial^\mu \eta^a + g f^{abc} (\partial^\mu \bar{\eta}^a) \eta^b A_\mu^c - \xi \bar{\eta}^a \eta^b [g T^a \langle \phi \rangle]_i [g T_{ij}^b (\langle \phi \rangle + \phi)_j] \\
 &= \overset{\text{VII}}{\partial_\mu \bar{\eta}^a \partial^\mu \eta^a} + \overset{\text{P}}{g f^{abc} (\partial^\mu \bar{\eta}^a) \eta^b A_\mu^c} - \xi \bar{\eta}^a \eta^b \overset{\text{VIII}}{[g T^a \langle \phi \rangle]_i} \overset{\text{IX}}{[g T_{ij}^b \langle \phi \rangle]_j} \\
 &\quad - \xi \bar{\eta}^a \eta^b [g T^a \langle \phi \rangle]_i (g T_{ij}^b \phi_j) \text{ ⑤}
 \end{aligned}$$

Hence, the full Lagrangian is:  $\mathcal{L}_{YM+Scalar} + \mathcal{L}_{GF} + \mathcal{L}_{ghost} = \mathcal{L}^{Quad} + \mathcal{L}^{Int}$

$$\begin{aligned} \mathcal{L}^{Quad} = & \frac{1}{2} \phi_i \left( \textcircled{I} -\partial^2 \delta_{ij} - \textcircled{II} M_{ij}^2 - \textcircled{III} \xi [gT^a \langle \phi \rangle]_i [gT^a \langle \phi \rangle]_j \right) \phi_j \\ & + \frac{1}{2} A_\mu^a \left( \textcircled{IV} \partial^2 g^{\mu\nu} \delta^{ab} - \left(1 - \frac{1}{\xi}\right) \textcircled{V} \partial^\mu \partial^\nu \delta^{ab} + \textcircled{VI} [gT^a \langle \phi \rangle]_i [gT^b \langle \phi \rangle]_i g^{\mu\nu} \right) A_\nu^b \\ & + \bar{\eta}^a \left( \textcircled{VII} -\partial^2 \delta^{ab} - \textcircled{VIII} \xi [gT^a \langle \phi \rangle]_i [gT^b \langle \phi \rangle]_i \right) \eta^b \end{aligned}$$

$$\begin{aligned} \mathcal{L}^{int} = & \textcircled{I} gf^{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c} - \frac{g^2}{4} \textcircled{II} f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} + \left( \text{Scalar cubic and quartic self interactions from } -V(\Phi) \right) \\ & + \textcircled{III} A^{\mu b} \partial_\mu \phi_i gT_{ij}^b \phi_j + \textcircled{IV} A_\mu^a A^{\mu b} [gT^a \langle \phi \rangle]_i gT_{ij}^b \phi_j + \frac{1}{2} \textcircled{V} A_\mu^a A^{\mu b} (gT^a \phi)_i (gT^b \phi)_i \\ & + \textcircled{VI} gf^{abc} (\partial^\mu \bar{\eta}^a) \eta^b A_\mu^c - \textcircled{VII} \xi [gT^a \langle \phi \rangle]_i \bar{\eta}^a \eta^b (gT_{ij}^b \phi_j) \end{aligned}$$

If  $V(\Phi)$  is a quartic polynomial in  $\Phi$ , we have:

$$\begin{aligned} V(\Phi) = & \frac{1}{2} \underbrace{\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j}}_{M_{ij}^2} \Big|_{\langle \phi \rangle} \phi_i \phi_j + \frac{1}{3!} \underbrace{\frac{\partial^3 V}{\partial \Phi_i \partial \Phi_j \partial \Phi_k}}_{\equiv C_{ijk}} \Big|_{\langle \phi \rangle} \phi_i \phi_j \phi_k + \frac{1}{4!} \underbrace{\frac{\partial^4 V}{\partial \Phi_i \partial \Phi_j \partial \Phi_k \partial \Phi_l}}_{\equiv C_{ijkl}} \Big|_{\langle \phi \rangle} \phi_i \phi_j \phi_k \phi_l \\ = & \frac{1}{2} M_{ij}^2 \phi_i \phi_j + \frac{1}{3!} C_{ijk} \phi_i \phi_j \phi_k + \frac{1}{4!} C_{ijkl} \phi_i \phi_j \phi_k \phi_l \end{aligned}$$

Clean up - Define:

Goldstone boson decay matrix  $F^a_i = gT_{ij}^a \langle \phi \rangle_j$

the gauge boson mass matrix,  $(m_A^2)^{ab} \equiv m_A^2 \langle \phi \rangle^{ab} = [gT^a \langle \phi \rangle]_i [gT^b \langle \phi \rangle]_i = FFT$

the pseudo-Goldstone mass matrix,  $(m_A^2)_{ij} \equiv m_A^2 \langle \phi \rangle_{ij} = [gT^a \langle \phi \rangle]_i [gT^a \langle \phi \rangle]_j = FTF$

$$\text{Then } [gT^a \langle \phi \rangle]_i gT_{ij}^b = \frac{1}{2} \frac{\partial (m_A^2)^{ab}}{\partial \langle \phi \rangle_j}$$

Covariant matrix (of the form  $vTv$ )

$$(gT^a \phi)_i (gT^b \phi)_i = \frac{1}{2} \frac{\partial^2 (m_A^2)^{ab}}{\partial \langle \phi \rangle_j \partial \langle \phi \rangle_k} \phi_j \phi_k$$

So, the final Lagrangian is  $\mathcal{L} = \mathcal{L}^{\text{Quad}} + \mathcal{L}^{\text{Int}}$  :

$$\mathcal{L}^{\text{Quad}} = \frac{1}{2} \phi_i \left( -\partial^2 \delta_{ij} - M_{ij}^2 - \xi (m_A^2)_{ij} \right) \phi_j$$

$$+ \frac{1}{2} A_\mu^\alpha \left( \partial^2 g^{\mu\nu} \delta^{ab} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \delta^{ab} + (m_A^2)^{ab} g^{\mu\nu} \right) A_\nu^b$$

$$+ \bar{\eta}^a \left( -\partial^2 \delta^{ab} - \xi (m_A^2)^{ab} \right) \eta^b$$

$$\mathcal{L}^{\text{Int}} = +g f^{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c} - \frac{g^2}{4} f^{abc} f^{ace} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} - \frac{1}{3!} c_{ijk} \phi_i \phi_j \phi_k$$

$$- \frac{1}{4!} c_{ijkl} \phi_i \phi_j \phi_k \phi_l + A^{\mu a} \partial_\mu \phi_i g T_{ij}^a \phi_j + \frac{1}{2} \frac{\partial (m_A^2)^{ab}}{\partial \langle \phi \rangle_i} A_\mu^\alpha A^{\mu b} \phi_i$$

$$+ \frac{1}{2} \frac{\partial^2 (m_A^2)^{ab}}{\partial \langle \phi \rangle_i \partial \langle \phi \rangle_j} A_\mu^\alpha A^{\mu b} \phi_i \phi_j + g f^{abc} (\partial^\mu \eta^a) \eta^b A_\mu^c - \frac{\xi}{2} \frac{\partial (m_A^2)^{ab}}{\partial \langle \phi \rangle_i} \bar{\eta}^a \eta^b \phi_i$$

This last term seems to remove the  $O_i$  belonging to each of the  $M_{ij}, (m_A)_{ij}$  subspaces. Together, they form  $1/p^2$  matrix proportional to the identity.

## Feynman Rules (for $G_{\text{Amp}}(-p, \dots, p)$ )

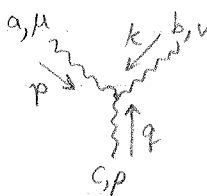
### Propagators

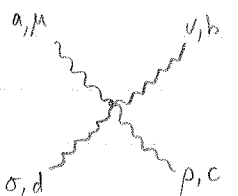
Scalar Fields  $i \xrightarrow{p} j = \frac{i}{p^2 - M_{ij}^2} + \frac{i}{p^2 - \xi(m_A^2)_{ij}} - \frac{\delta_{ij}}{p^2}$  (form only valid at tree-level minimum)

Gauge Fields  $a \text{---} b = \frac{-i}{p^2 - (m_A^2)^{ab} + i\epsilon} \left( g^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2 - \xi(m_A^2)^{ab} + i\epsilon} \right)$

Ghost Fields  $a \dots b = \frac{i}{p^2 - \xi(m_A^2)^{ab} + i\epsilon}$

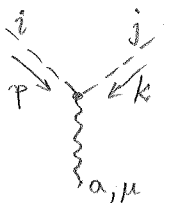
### Vertices

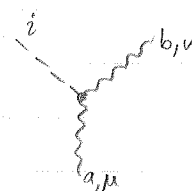
  $= g f^{abc} (g_{\mu\nu}(k-p)_\rho + g_{\nu\rho}(q-k)_\mu + g_{\rho\mu}(p-q)_\nu)$

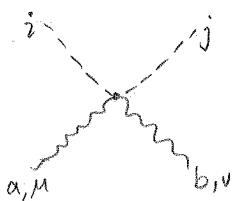
  $= -i g^2 [ f^{abe} f^{ecd} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{ebd} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{ebc} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}) ]$

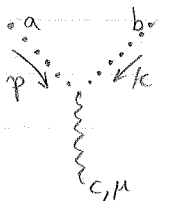
  $= -i C_{ijk}$

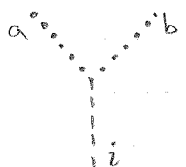
  $= -i C_{ijkl}$

  $= g T_{ij}^a (p-k)$

  $= i \frac{\partial(m_A^2)^{ab}}{\partial\langle\phi\rangle_i} g^{\mu\nu}$

  $= 2i \frac{\partial^2(m_A^2)^{ab}}{\partial\langle\phi\rangle_i \partial\langle\phi\rangle_j} g^{\mu\nu}$

  $= -\frac{1}{2} g f^{abc} (p-k)_\mu$

  $= -\frac{i\xi}{2} \frac{\partial(m_A^2)^{ab}}{\partial\langle\phi\rangle_i}$

Multiply by (-1) for each Ghost loop

Using complex scalar fields, there is an  $i$  in this Feynman rule  $\rightarrow$