Obtaining generators for Real representation

Generically, transformation law for a complex vector $\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \\ \vdots\end{pmatrix}$ is

$$\Phi \longrightarrow \left(e^{i\alpha^a T^a}\right)\Phi. \qquad \begin{array}{l} T^a \in \text{Lie Algebra} \\ \Phi \equiv \text{complex vector}\end{array}$$

This i mixes up the real and imaginary components of Φ.

So, to move to representation $\Phi = \begin{pmatrix}\phi_1 \\ \phi_2 \\ \vdots\end{pmatrix}$, use matrix representation $i = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$, doubling size of matrices, T^a.

[not $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$]

Then, $T^a_{Re} \equiv i \otimes T^a$, and $\phi \to \left(e^{\alpha^a T^a_{Re}}\right)\phi$ ← column vector of real components

① First, multiply all generators by i.

Gives "my" real representation ② Then, replace $i \to \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$, $1 \to \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$ $\Rightarrow$ Real, antisymmetric generators

Example

$SU(2)\times U(1)$ Higgs doublet: $H = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix}$

Fundamental rep: $T^a = \left\{\frac{1}{2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}, \begin{pmatrix}Y & 0 \\ 0 & Y\end{pmatrix}\right\}$.

↖ hypercharge $Y = 1/2$

Then, to get real representation:

① multiply by i $= \left\{\frac{1}{2}\begin{pmatrix}0 & i \\ i & 0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}i & 0 \\ 0 & -i\end{pmatrix}, \begin{pmatrix}iY & \\ & iY\end{pmatrix}\right\}$

② replace $i \to \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$ and $1 \to \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$.

My real representation

$$T^a_{Re} = \left\{ \frac{1}{2}\left(\begin{array}{cc|cc} & & & -1 \\ & & 1 & \\ \hline & -1 & & \\ 1 & & & \end{array}\right), \frac{1}{2}\left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline -1 & & & \\ & -1 & & \end{array}\right), \frac{1}{2}\left(\begin{array}{cc|cc} & -1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right), \left(\begin{array}{cc|cc} & -Y & & \\ Y & & & \\ \hline & & & -Y \\ & & Y & \end{array}\right) \right\}$$

③ For standard real representation (pure-imaginary, antisymmetric $\Rightarrow$ hermitian) multiply T^a_{Re} by $-i$:

standard real rep:

$$T^a_{Im} = \left\{ \frac{1}{2}\left(\begin{array}{cc|cc} & & & i \\ & & -i & \\ \hline & i & & \\ -i & & & \end{array}\right), \frac{1}{2}\left(\begin{array}{cc|cc} & & -i & \\ & & & -i \\ \hline i & & & \\ & i & & \end{array}\right), \frac{1}{2}\left(\begin{array}{cc|cc} & i & & \\ -i & & & \\ \hline & & & -i \\ & & i & \end{array}\right), \left(\begin{array}{cc|cc} & iY & & \\ -iY & & & \\ \hline & & & iY \\ & & -iY & \end{array}\right) \right\},$$

which restores "i" in transformation law: $\phi \to e^{i\alpha^a T^a_{Im}}\phi$.

To go to charge-eigenstate basis $(\phi^+, \phi^-, \phi^0, \phi^{0*})$, use $U = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} 1 & i & & \\ 1 & -i & & \\ \hline & & 1 & i \\ & & 1 & -i \end{array}\right)$:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & & \\ 1 & -i & & \\ & & 1 & i \\ & & 1 & -i \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_1 - i\phi_2 \\ \phi_3 + i\phi_4 \\ \phi_3 - i\phi_4 \end{pmatrix} = \begin{pmatrix} \phi^+ \\ \phi^- \\ \phi^0 \\ \phi^{0*} \end{pmatrix}$$

Notice, in this basis, T^3_{Im} is diagonal:

$$U T^3_{Im} U^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i & & \\ 1 & -i & & \\ & & 1 & i \\ & & 1 & -i \end{pmatrix}\left(\begin{array}{cc|cc} & i & & \\ -i & & & \\ \hline & & & -i \\ & & i & \end{array}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & & \\ -i & i & & \\ & & 1 & 1 \\ & & -i & i \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$$

$\Rightarrow$ ϕ^+ and ϕ^{0*} are isospin "up"

ϕ^- and ϕ^0 are isospin "down".

<u>Warning!</u> Moving to real representation <u>doubles</u> the size of the representation matrices.

Group theory traces $T_R\,\delta^{ab} = \mathrm{Tr}[T^a_R, T^b_R]$ will double.

typically this will be accounted for by a compensating change elsewhere, say in the path integral, where you get $\det(\mathcal{O})^{-1/2}$ instead of $\det(\mathcal{O})^{-1}$.