Finite Transformation Law for Quantum Fields in a Background

$$\underline{\delta\phi_i(x) = \alpha^a T^a_{ij}(\bar\phi+\phi)_j} \,,$$

notation: $\bar\phi \equiv \langle\phi\rangle$

$T^a_{ij} \equiv$ real, antisymmetric

Obtain finite transformation law by making a large number of infinitessimal ones:

$$\phi \longrightarrow \phi + \alpha^a T^a(\bar\phi+\phi)$$

$$= \alpha^a T^a\bar\phi + \phi + \alpha^a T^a \phi$$

$$\longrightarrow \alpha^a T^a\bar\phi + \phi + \alpha^a T^a(\bar\phi+\phi) + \alpha^a T^a\phi + \alpha^a T^a \alpha^b T^b(\bar\phi+\phi)$$

$$= 2\alpha^a T^a\bar\phi + \alpha^a T^a\alpha^b T^b\bar\phi + \phi + 2\alpha^a T^a\phi + \alpha^a T^a\alpha^b T^b\phi$$

$\vdots$ (scale $\alpha \to \frac{\alpha}{N}$, $N \equiv$ number of infinitessimal transformations)

$$= \frac{\alpha^a T^a}{N}\left(1+\frac{\alpha^b T^b}{N}\right)^N\bar\phi + \left(1+\frac{\alpha^a T^a}{N}\right)^N\phi$$

Take $N\to\infty$ limit:

$$\boxed{\begin{aligned}\phi &\longrightarrow \left(e^{\alpha^a T^a}-1\right)\bar\phi + e^{\alpha^a T^a}\phi \\ &= e^{\alpha^a T^a}(\bar\phi+\phi) - \bar\phi\end{aligned}}$$

i.e. $(\phi+\bar\phi) \to e^{\alpha^a T^a}(\bar\phi+\phi)$ as expected.