Identities

Symmetry properties of $V(\Phi)$ impose constraints on its expansion coefficients, M_{ij}^2, c_{ijk} & c_{ijkl}:

If $V(\Phi)$ is invariant under some group (not necessarily simple), then:

$$V'(\Phi') = V(\Phi) \quad \text{where} \quad \Phi_i' = \left(e^{\overset{\text{absorb "}i\text{"}}{i\alpha^a g T^a_{Im}}}\right)_{ij} \Phi_j \simeq (1 + \alpha^a g T^a_{Re})_{ij} \Phi_j$$

$\alpha^a \equiv$ transformation parameter

$gT^a \equiv$ generator

Then

$$V'(\Phi') = V'((1 + \alpha^a g T^a)\Phi)$$

$$\simeq \cancel{V(\Phi)} + \alpha^a (gT^a\Phi)_i \frac{\partial V}{\partial \Phi_i} = \cancel{V(\Phi)}$$

$$\Rightarrow \boxed{(gT^a\Phi)_i \frac{\partial V}{\partial \Phi_i} = 0} \quad \text{since eqn true for any } \alpha^a.$$

Now differentiate this with respect to Φ_j

$$\boxed{g T^a_{ij} \frac{\partial V}{\partial \Phi_i} + (gT^a\Phi)_i \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} = 0} \quad \text{or} \quad gT^a_{ji}\frac{\partial V}{\partial \Phi_j} + \frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j}(gT^a\Phi)_j = 0$$

(used symm of $\frac{d}{d\Phi}$)

Note: $\frac{\partial^2 V}{\partial \Phi_i \partial \Phi_j} = M_{ij}^2(\Phi)$ is the shifted mass matrix when all scalars are shifted.

When this identity is evaluated at the tree level minimum, $\langle \Phi \rangle = v_0$, we have

$$\left.\frac{\partial V}{\partial \Phi_j}\right|_{\langle \phi \rangle} = 0 \quad \Rightarrow \quad \boxed{\begin{gathered}(gT^a\langle\phi\rangle)_i M_{ij}^2 = 0 \\ \text{or} \\ (m_A^2)_{ki} M_{ij}^2 = 0\end{gathered}}$$

where M_{ij}^2 is the tree-level mass matrix for scalar fields.

This means at the tree-level minimum,

$[m_A^2, M^2] = 0$ and m_A^2 & M^2 are simultaneously diagonalizable. (VERY IMPORTANT)

Note that $m_A^2(\phi)^{ab} = (gT^a\phi)_i (gT^b\phi)_i$ and $m_A^2(\phi)_{ij} = (gT^a\phi)_i (gT^a\phi)_j$ have the same non-zero eigenvalues — even as functions of ϕ.

<u>Proof</u>: Let ω^b be an e-vector of $m_A^2(\phi)^{ab}$ with e-value λ.

$$m_A^2(\phi)^{ab}\,\omega^b = (gT^a\phi)_i (gT^b\phi)_i\,\omega^b = \lambda\,\omega^a$$

Now multiply by $(gT^a\phi)_j$ and sum over a.

$$\underbrace{(gT^a\phi)_j (gT^a\phi)_i}_{m_A^2(\phi)_{ji}} \underbrace{(gT^b\phi)_i\,\omega^b}_{\text{Define: } = \mathcal{N}\omega_i} = \lambda \underbrace{(gT^a\phi)_j\,\omega^a}_{= \mathcal{N}\omega_j}$$

$$m_A^2(\phi)_{ji}\,\mathcal{N}\omega_i = \lambda\,\mathcal{N}\omega_j$$

or $$\underline{m_A^2(\phi)_{ij}\,\mathcal{N}\omega_j = \lambda\,\mathcal{N}\omega_i} \qquad \blacksquare$$

To get the normalization for ω_j (the e-vector of $m_A^2(\phi)_{ij}$), suppose ω^a is normalized: $\omega^a\omega^a = 1$

Then $$\omega_i\omega_i = \frac{1}{\mathcal{N}^2}(gT^a\phi)_i\,\omega^a\,(gT^b\phi)_i\,\omega^b$$

$$= \frac{1}{\mathcal{N}^2}\underbrace{(gT^a\phi)_i (gT^b\phi)_i\,\omega^b}_{m_A^2(\phi)^{ab}\omega^b = \lambda\omega^a}\omega^a$$

$$= \frac{1}{\mathcal{N}^2}\lambda\underbrace{\omega^a\omega^a}_{1} = \frac{1}{\mathcal{N}^2}\lambda. = 1$$

Then $\Rightarrow$ $$\mathcal{N} = \frac{1}{\sqrt{\lambda}}$$

So, if ω^b is an e-vector of $m_A^2(\phi)^{ab}$ with e-value λ, then $\omega_j = \frac{1}{\sqrt{\lambda}}(gT^a\phi)_j\,\omega^a$ is an e-vector of $m_A^2(\phi)_{ij}$ with the same e-value.

<u>NOTE</u>: Nothing can be said about e-vectors with zero e-values. In fact $(m_A^2)^{ab}$ & $(m_A^2)_{ij}$ may have differing number of zero e-values.