Non-abelian case – REPEAT USING COMPLEX REPRESENTATION

– This time, leave scalars in complex representation of general gauge group

Gauge transformation rule for $\Phi_i(x)$

$$\Phi_i(x) \longrightarrow \left(\mathbb{1}_{ij} + i\alpha^a(x) T^a_{ij}\right)\Phi_j(x)$$

Hermitian generators

Covariant derivative:

$$(D_\mu)_{ij}\Phi_j = \left(\delta_{ij}\partial_\mu + i g T^a_{ij} A^a_\mu\right)\Phi_j$$

Gauge field transformation rule:

$$A^a_\mu \longrightarrow A^a_\mu - \partial_\mu \frac{\alpha^a}{g} - f^{abc}\alpha^b A^c_\mu = A^a_\mu - \left(D^{ab}_\mu \frac{\alpha^b}{g}\right)$$

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Lagrangian:

$$\mathcal{L} = |D_\mu \Phi|^2 - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - V(\Phi)$$

$$= \partial_\mu \Phi^\dagger \partial^\mu \Phi + \left(i g A^a_\mu \partial^\mu \Phi^*_i T^a_{ij}\Phi_j + \text{h.c.}\right)$$

$$+ A^a_\mu A^{\mu b}(gT^a\Phi)^*_i (gT^b\Phi)_i - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - V(\Phi)$$

→ Suppose $V(\Phi)$ is such that $\Phi$ acquires non-zero vacuum expectation value in some of its components

Write $\Phi_i(x) = \langle\phi\rangle_i + \phi_i(x)$
$$\begin{cases} \delta\phi_i = i\alpha^a T^a_{ij}(\langle\phi\rangle + \phi)_j \\ \delta\langle\phi\rangle_i = 0 \end{cases}$$

Space spanned by non-zero directions $T^a\langle\phi\rangle_i$ are the Goldstone bosons.

Retain terms quadratic in fields (higher order terms give rise to interactions)

$$\mathcal{L}^{(2)} = \frac{1}{2} A^a_\mu \left( \partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu \right) A^a_\nu + \partial_\mu \phi_i^* \partial^\mu \phi_i - \frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\bigg|_{\phi = \langle \phi \rangle}$$

$$+ \underbrace{\left( i A^a_\mu \partial^\mu \phi_i^* (g T^a_{ij} \langle \phi \rangle_j) + h.c. \right)}_{\substack{(\partial . A)\phi \text{ terms to be removed} \\ \text{by fixing gauge}}} + \underbrace{(g T^a_{ij} \langle \phi \rangle_j)(g T^b_{ik} \langle \phi \rangle_k)^* A^a_\mu A^{\mu b}}_{\substack{\text{Gauge boson mass terms.} \\ \text{[symmetric part is the mass matrix].}}} + \ldots$$

When quantizing system, $\mathcal{L}_{GF} = \frac{-1}{2\xi} (\mathcal{F}^a)^2$ is added.

't Hooft $R_\xi$ gauge: $\mathcal{F}^a = (\partial . A^a - i \xi \phi_i^* (g T^a \langle \phi \rangle)_i + h.c.)$

$$\mathcal{L}_{GF} = \frac{-1}{2\xi} (\partial_\mu A^{\mu a})^2 - \frac{1}{2\xi} 2 \left( -\partial . A^a \, i \xi (g T^a \langle \phi \rangle)_i \phi_i + h.c. \right)$$

$$- \frac{\xi}{2} \left( -i \phi_i^* (g T^a_{ij} \langle \phi \rangle_j) + h.c. \right)^2$$

First and second terms $\rightarrow$ integrate by parts; third term factor out $-i$

$$= \frac{-1}{2\xi} A^a_\mu \partial^\mu \partial^\nu A^a_\nu - \left( i A^a_\mu \partial^\mu \phi_i^* (g T^a \langle \phi \rangle)_i + h.c. \right) \longleftarrow \text{cancels against } (\partial . A)\phi \text{ term.}$$

$$+ \frac{\xi}{2} \left( \phi_i^* (g T^a \langle \phi \rangle)_i - (g T^a \langle \phi \rangle)_i^* \phi_i \right)^2 \longleftarrow \text{Gives Goldstone bosons gauge dep. masses.}$$

Next, work on Fadeev-Popov determinant:

$$\mathcal{F}^a(\underbrace{A', \phi'}_{\text{gauge transformed}}) = \partial^\mu A'^a_\mu - i \xi \phi_i'^* (g T^a \langle \phi \rangle)_i + h.c.$$

$$= \partial^\mu A_\mu - \partial^2 \frac{\alpha^a}{g} - f^{abc} (\partial^\mu \alpha^b) A^c_\mu - f^{abc} \alpha^b \partial^\mu A^c_\mu$$

$$- i \xi \phi_i^\dagger (g T^a \langle \phi \rangle)_i - \xi \left( \alpha^b (\langle \phi \rangle + \phi)_i^* T^b_{ij} \right) (g T^a \langle \phi \rangle)_j + h.c.$$

Now use:

$$\frac{\delta \mathcal{F}'^a(x)}{\delta \alpha^b(y)} = \frac{\partial \mathcal{F}'^a}{\partial \alpha^b}\,\delta^{(4)}(x-y) + \frac{\partial \mathcal{F}'^a}{\partial(\partial^\mu \alpha^b)}\,\partial^\mu \delta^{(4)}(x-y) + \frac{\partial \mathcal{F}'^a}{\partial(\partial^\mu\partial^\nu \alpha^b)}\,\partial^\mu\partial^\nu \delta^{(4)}(x-y)$$

$$= \left[ -f^{abc}\,\partial^\mu A^c_\mu - \left( \xi\,(\langle\phi\rangle+\phi)^*_i\, T^b_{ij}\,(gT^a\langle\phi\rangle)_j + \text{h.c.} \right) \right] \delta^{(4)}(x-y)$$

$$-f^{abc} A^c_\mu\, \partial^\mu \delta^{(4)}(x-y) - \frac{1}{g}\,\partial^2 \delta^{(4)}(x-y)$$

$$= \Big[ \underbrace{-\frac{\delta^{ab}}{g}\,\partial^2 - f^{abc}\,\partial^\mu A^c_\mu - f^{abc} A^c_\mu\,\partial^\mu}\ \longleftarrow\ -\partial^\mu \frac{D^{ab}_\mu}{g}$$

$$- \left( \xi\,(\langle\phi\rangle+\phi)^*_i\, T^b_{ij}\,(gT^a\langle\phi\rangle)_j + \text{h.c.} \right) \Big]\, \delta^{(4)}(x-y)$$

Then $\det\left( \dfrac{\delta \mathcal{F}'^a(x)}{\delta \alpha^b(y)} \right) \longrightarrow \displaystyle\int \mathcal{D}\bar\eta\, \mathcal{D}\eta\; e^{\,i\int d^4x \left[ -\bar\eta^a \left( \partial^\mu \frac{D^{ab}_\mu}{g} + \xi(\langle\phi\rangle+\phi)^*_i T^b_{ij} (gT^a\langle\phi\rangle)_j + \text{h.c.} \right) \eta^b \right]}$

So that

$$\mathcal{L}_{\text{Ghost}} = -\bar\eta^a \Big[ \frac{1}{g}\,\partial_\mu D^{\mu\,ab} + \xi\,(\langle\phi\rangle+\phi)^*_i\, T^b_{ij}\,(gT^a\langle\phi\rangle)_j + \xi\,(g\langle\phi\rangle^* T^a)_i\, T^b_{ij}\,(\langle\phi\rangle+\phi)$$

$$= -\frac{1}{g}\,\bar\eta^a \Big[ \partial^2 \delta^{ab} + g f^{abc}\,\partial_\mu (A^{\mu c}\,\cdot\,) + \xi\big[ (\langle\phi\rangle+\phi)^* T^b g \big]_i\, (gT^a\langle\phi\rangle)_i + \text{h.c.} \Big]\, \eta^b$$

absorb    Integrate by parts    swap

$$= +\partial_\mu \bar\eta^a\, \partial^\mu \eta^a + g f^{abc} (\partial_\mu \bar\eta^a)\, \eta^b A^{\mu c}$$

$$-\bar\eta^a\, \underbrace{\xi \Big[ (gT^a\langle\phi\rangle)_i\, (gT^b\langle\phi\rangle)^*_i + (gT^a\langle\phi\rangle)^*_i\, (gT^b\langle\phi\rangle)_i \Big]}_{\text{this whole quantity is ghost mass matrix.}}\, \eta^b$$

$$-\bar\eta^a\, \xi \Big[ (gT^a\langle\phi\rangle)_i\, (gT^b\phi)^*_i + (gT^a\langle\phi\rangle)^*_i\, (gT^b\phi)_i \Big]\, \eta^b$$

Thus, adding everything together,

$$\mathcal{L}^{\text{quad}} = \partial_\mu \phi_i^* \partial^\mu \phi_i - \underbrace{\frac{\partial^2 V}{\partial \phi_i^\dagger \partial \phi_j}}_{\text{physical masses}} \phi_i^* \phi_j + \frac{\xi}{2} \big( \underbrace{\phi_i^* (gT^a \langle \phi \rangle)_i + (gT^a \langle \phi \rangle)_i^* \phi_i}_{\text{Goldstone mass terms}} \big)^2$$

$$+ \frac{1}{2} A_\mu^a \Big( \partial^2 g^{\mu\nu} \delta^{ab} - \big(1 - \tfrac{1}{\xi}\big) \partial^\mu \partial^\nu \delta^{ab} + (gT^{[a}_{ij} \langle \phi \rangle_j)(gT^{b]}_{ik} \langle \phi \rangle_k)^* g^{\mu\nu} \Big) A_\nu^b$$

$$+ \partial_\mu \bar{\eta}^a \partial^\mu \eta^b - \bar{\eta}^a \underbrace{\xi \Big[ (gT^a \langle \phi \rangle)_i (gT^b \langle \phi \rangle)_i^* + (gT^a \langle \phi \rangle_i)^* (gT^b \langle \phi \rangle)_i \Big]}_{\text{F.P. ghost mass matrix}} \eta^b$$

$$\mathcal{L}^{\text{int}} = + g f^{abc} \partial_\mu A_\nu^a A^{\mu b} A^{\nu c} - \frac{g^2}{4} f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} + \begin{pmatrix} \text{cubic + quartic} \\ \text{potential terms} \end{pmatrix}$$

$$+ \begin{pmatrix} \text{scalar QED like} \\ \phi\text{-}A\text{-}A \text{ coupling} \end{pmatrix} + \begin{pmatrix} \text{scalar QED like} \\ A\text{-}A\text{-}\phi\text{-}\phi \text{ coupling} \end{pmatrix} + \begin{pmatrix} \langle \phi \rangle\text{-dep.} \\ \phi\text{-}A\text{-}A \text{ coupling} \end{pmatrix}$$

$$+ g f^{abc} (\partial^\mu \bar{\eta}^a) \eta^b A_\mu^c$$

$$- \bar{\eta}^a \, \xi \Big[ (gT^a \langle \phi \rangle)_i (gT^b \phi)_i^* + \underbrace{(gT^a \langle \phi \rangle)_i^* (gT^b \phi)_i}_{\substack{\text{this term is c.c. of} \\ \text{first term.}}} \Big] \eta^b$$