Canonical quantization of massive spin-1 field in Fermi-ξ gauges

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial\cdot A)^2 - A_\mu J^\mu$$

conserved current

write as Feynman + extra: $\frac{1}{2\xi}(\partial\cdot A)^2 = \frac{1}{2}(\partial\cdot A)^2 + \frac{1}{2}\left(\frac{1}{\xi}-1\right)(\partial\cdot A)^2$

$$= \frac{1}{2}(\partial\cdot A)^2 + \frac{1}{2}\left(\frac{1-\xi}{\xi}\right)(\partial\cdot A)^2$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(\partial\cdot A)^2 + \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{2}\frac{1-\xi}{\xi}(\partial\cdot A)^2 - A_\mu J^\mu$$

$$= \underbrace{-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu}_{\text{Fermi type kinetic term}} + \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{2}\frac{1-\xi}{\xi}(\partial\cdot A)^2 + \left(\text{total divergence}\right) - A_\mu J^\mu$$

Equation of motion

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0$$

$$(m^2 A^\nu - J^\nu) - \partial_\mu\left[-\partial^\mu A^\nu - g^{\mu\nu}\left(\frac{1-\xi}{\xi}\right)\partial\cdot A\right] = 0$$

$$\therefore \boxed{(\Box + m^2)A^\nu - \left(1-\frac{1}{\xi}\right)\partial^\nu \partial\cdot A = J^\nu}$$

When solving EOM, useful to take 4-divergence to isolate scalar polarization: $-\frac{1}{\xi}\partial\cdot A \equiv B$

$$(\Box + m^2)\underbrace{\partial\cdot A}_{-\xi B} - \left(1-\frac{1}{\xi}\right)\Box \underbrace{\partial\cdot A}_{-\xi B} = \underbrace{\partial\cdot J}_{=0}$$ (assume conserved current)

$$-\xi(\Box + m^2) - (1-\xi)\Box B = 0$$

$$-(\Box + \xi m^2)B = 0 \quad \Rightarrow \quad \boxed{\text{scalar polarization has mass } \sqrt{\xi}\, m}$$

Solution to equation of motion.

$$(\Box^2 + m^2) A_\mu - (1 - \tfrac{1}{\xi}) \partial_\mu \partial\cdot A = 0$$

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{\lambda = \{\pm1, 0\}} \left[\epsilon_\mu^{[\lambda]}(\vec{p})\, a_{\vec{p},\lambda}\, e^{-ip\cdot x} + \epsilon_\mu^{*[\lambda]}(\vec{p})\, \hat{a}^*_{\vec{p},\lambda}\, e^{ip\cdot x} \right]$$

$$+ \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p(\xi)}} \left[\epsilon_\mu^{[sc]}(p(\xi))\, a_{\vec{p},sc}\, e^{-ip(\xi)\cdot x} + \epsilon_\mu^{*[sc]}(p(\xi))\, a^*_{\vec{p},sc}\, e^{ip(\xi)\cdot x} \right]$$

$$\omega_p(\xi) = \sqrt{\vec{p}^2 + \xi m^2} \qquad p^\mu(\xi) = (\omega_{\vec{p}}(\xi), \vec{p})$$

$$\epsilon_\mu^{[sc]} = p_\mu(\xi)/m \leftarrow \text{no } \xi \text{ in denominator}$$

Normalization of scalar polarization:

$$\epsilon^{[sc]*} \cdot \epsilon^{[sc]} = \frac{p^2(\xi)}{m^2} = \xi.$$

orthogonality in rest frame

$$\epsilon_\mu^{[sc]} = (\sqrt{\xi}\, m; 0)$$
$$\epsilon_\mu^{[0,\pm1]} = (0; \vec{\varepsilon})$$

still orthogonal

- Scalar fluctuations have mass (squared) = ξm^2
- Scalar polarization is gauge dependent: $\epsilon_\mu^{[sc]} = \sqrt{\frac{\vec{p}^2}{m^2} + \xi}$

Orthonormality

$$\epsilon^{[s]*}(\vec{p}, \xi) \cdot \epsilon^{[s']}(\vec{p}, \xi) = \text{``}g_{ss'}\text{''} - \delta_{0s}\delta_{0s'}(1 - \xi).$$

Completeness

$$\sum_{s = \{\pm1, 0, \text{scalar}\}} \epsilon_\mu^{*[s]}(\vec{p}, \xi)\, \epsilon_\nu^{[s]}(\vec{p}, \xi) \left[\text{``}g_{ss}\text{''} - \delta_{0s}\delta_{0s}(1 - \tfrac{1}{\xi}) \right] = + g_{\mu\nu}$$

check: contract with ϵ^μ & $\epsilon^{\nu *}$

$$\sum_{s=\{\pm1, 0, \text{scalar}\}}' \left(g_{ss'} - \delta_{0s}\delta_{0s'}(1-\xi)\right)\left(g_{ss'} - \delta_{0s}\delta_{0s'}(1-\xi)\right) \left[g_{ss} - \delta_{0s}\delta_{0s}(1 - \tfrac{1}{\xi}) \right]$$

✓ checked by computer

$$\stackrel{?}{=} \left[g_{s's'} - \delta_{0s'}\delta_{0s'}(1-\xi) \right]$$

Hamiltonian

Conjugate momentum:

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\dot{A}^\mu)} = -\dot{A}_\mu - \left(\frac{1-\xi}{\xi}\right) g_{\mu 0}\, \partial\cdot A$$

$$= -\dot{A}_\mu - \left(\frac{1-\xi}{\xi}\right) g_{\mu 0} \left(\dot{A}_0 - \vec{\nabla}\cdot\vec{A}\right)$$

Invert to obtain $\dot{A}_\mu$ in terms of π_μ

$$\pi_0 = -\dot{A}_0 - \frac{1-\xi}{\xi}\left(\dot{A}_0 - \vec{\nabla}\cdot\vec{A}\right) \qquad \vec{\pi} = -\dot{\vec{A}}$$

$$= -\left(\cancel{1} + \frac{1-\cancel{\xi}}{\xi}\right)\dot{A}_0 - \frac{1-\xi}{\xi}\,\vec{\nabla}\cdot\vec{A}$$

$$\Rightarrow \dot{A}_0 = -\xi\pi_0 - (1-\xi)\,\vec{\nabla}\cdot\vec{A} \qquad \Rightarrow \dot{\vec{A}} = -\vec{\pi}$$

$$= -\pi_0 + (1-\xi)\left(\pi_0 - \vec{\nabla}\cdot\vec{A}\right)$$

Combine $\dot{A}_0$ and $\dot{\vec{A}}$

$$\boxed{\dot{A}_\mu = -\pi_\mu + g_{\mu 0}(1-\xi)\left(\pi_0 - \vec{\nabla}\cdot\vec{A}\right)}$$

Construct Hamiltonian:

$$\mathcal{L} = -\frac{1}{2}\dot{A}_\mu\dot{A}^\mu + \frac{1}{2}(\vec{\nabla}A_\mu)\cdot(\vec{\nabla}A^\mu) + \frac{1}{2}m^2 A_\mu A^\mu + \frac{1-\xi}{2\xi}\left(\dot{A}_0 + \vec{\nabla}\cdot\vec{A}\right)^2 - A_\mu J^\mu$$

$$\mathcal{H} = \pi_\mu \dot{A}^\mu - \mathcal{L}$$

$$= -\pi_\mu\pi^\mu + \pi_0(1-\xi)\left(\pi_0 - \vec{\nabla}\cdot\vec{A}\right)$$

$$+ \frac{1}{2}\left(-\pi_\mu + g_{\mu 0}(1-\xi)\left(\pi_0 - \vec{\nabla}\cdot\vec{A}\right)\right)\left(-\pi^\mu + g^{\mu 0}(1-\xi)\left(\pi_0 - \vec{\nabla}\cdot\vec{A}\right)\right)$$

$$- \frac{1}{2}(\vec{\nabla}A_\mu)\cdot(\vec{\nabla}A^\mu) - \frac{1}{2}m^2 A_\mu A^\mu + \frac{1-\xi}{2\xi}\left(-\xi\pi_0 \underbrace{-(1-\xi)\,\vec{\nabla}\cdot A + \vec{\nabla}\cdot\vec{A}}_{\text{cancel}}\right)^2$$

$$+ A_\mu J^\mu$$

Expand out middle line.

Factor out ξ from final term of 3rd line

$$= -\pi_\mu \pi^\mu \cancel{+ \pi^0 (1-\xi)(\pi^0 - \vec{\nabla}\cdot\vec{A})}$$

$$+ \left[\frac{1}{2}\pi_\mu \pi^\mu \cancel{- \pi_0 (1-\xi)(\pi^0 - \vec{\nabla}\cdot\vec{A})} + \frac{1}{2}(1-\xi)^2(\pi_0 - \vec{\nabla}\cdot\vec{A})^2\right]$$

$$- \frac{1}{2}(\vec{\nabla}A_\mu)\cdot(\vec{\nabla}A^\mu) - \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}\xi(1-\xi)(\pi_0 - \vec{\nabla}\cdot\vec{A})^2$$

$$+ A_\mu J^\mu$$

combine final terms of 2nd & 3rd lines:

$$\frac{1}{2}(1-2\xi+\xi^2) + \frac{1}{2}(\xi - \xi^2) = \frac{1}{2}(1-\xi)$$

$$\boxed{\mathcal{H} = -\frac{1}{2}\pi^\mu \pi_\mu - \frac{1}{2}(\vec{\nabla}A_\mu)\cdot(\vec{\nabla}A^\mu) - \frac{1}{2}m^2 A_\mu A^\mu + \frac{1}{2}(1-\xi)(\pi_0 - \vec{\nabla}\cdot\vec{A})^2 + A_\mu J^\mu}$$

Poisson bracket:

$$\left\{A^\mu(\vec{x}),\ \pi_\nu(\vec{x}\,')\right\}_{PB} = \int d^4z \left(\frac{\delta A^\mu(\vec{x})}{\delta A_\rho(\vec{z})} g^{\rho\sigma} \frac{\delta \pi_\nu(\vec{x}\,')}{\delta \pi_\sigma(\vec{z})} - \frac{\delta A^\mu(\vec{x})}{\delta \pi_\sigma(\vec{z})} g^{\rho\sigma} \frac{\delta \pi_\nu(\vec{x}\,')}{\delta A_\rho(\vec{z})}\right)$$

$$= \delta^{(3)}(\vec{x}-\vec{x}\,')\,\delta^\mu{}_\nu$$

$$\left\{A^\mu(\vec{x}),\ A_\nu(\vec{x}\,')\right\}_{PB} = \left\{\pi^\mu(\vec{x}),\ \pi_\nu(\vec{x})\right\}_{P.B} = 0$$

Canonical commutation relations

$$\left[\hat{A}^\mu(\vec{x},t),\ \hat{\pi}_\nu(\vec{x}\,',t)\right] = i\,\delta^\mu{}_\nu\,\delta^{(3)}(\vec{x}-\vec{x}\,')$$

equal time

$$\left[\hat{A}^\mu(\vec{x},t),\ A_\nu(\vec{x}\,',t)\right] = \left[\hat{\pi}^\mu(\vec{x},t),\ \hat{\pi}_\nu(\vec{x},t)\right] = 0$$

In plane wave expansion,
mode coefficients become operators $a_{\vec{p},\lambda} \longrightarrow \hat{a}_{\vec{p},\lambda}$

$$a^*_{\vec{p},\lambda} \longrightarrow \hat{a}^\dagger_{\vec{p},\lambda}.$$

Algebra to be consistent with CCR.