## Background Field Gauge

As before, put all scalars in the real representation — not as crucial, but convenient.

Then,

$$\mathcal{L} = \frac{1}{2}(D_\mu\phi)(D^\mu\phi) - V(\phi) - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + j^a_\mu A^{\mu a}$$

Recall: $D_\mu = \partial_\mu + A^a_\mu g T^a_{Re}$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$

Perform shift: $A^a_\mu \longrightarrow \bar{A}^a_\mu + A^a_\mu$

Classical, background field — Quantum fluctuations (to be integrated over)

> Under a gauge transformation of the quantum fields with respect to the background fields,
>
> $A^a_\mu \longrightarrow A^a_\mu - \left(\bar{D}^{ab}_\mu \frac{\alpha^b}{g}\right)$

Then, $F^a_{\mu\nu} \longrightarrow \partial_\mu(\bar{A}^a_\nu + A^a_\nu) - \partial_\nu(\bar{A}^a_\mu + A^a_\mu) - g f^{abc}(\bar{A}^b_\mu + A^b_\mu)(\bar{A}^c_\nu + A^c_\nu)$

$$= \partial_\mu \bar{A}^a_\nu - \partial_\nu \bar{A}^a_\mu - g f^{abc}\bar{A}^b_\mu \bar{A}^c_\nu$$

$$+ \underbrace{\partial_\mu A^a_\nu - g f^{abc} A^b_\mu \bar{A}^c_\nu} - \underbrace{\partial_\nu A^a_\mu - g f^{abc}\bar{A}^c_\nu A^b_\mu} - g f^{abc} A^b_\mu A^c_\nu$$

$$= \bar{F}^a_{\mu\nu} + (\bar{D}_\mu A_\nu)^a - (\bar{D}_\nu A_\mu)^a - g f^{abc} A^b_\mu A^c_\nu$$

where $\bar{D}^{ab}_\mu = \partial_\mu \delta^{ab} + g f^{abc}\bar{A}^c_\mu$ is the covariant derivative in a classical, background (gauge field).

c.f. the geometric cov. derivative, $\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\rho\mu} v^\rho$ of General Relativity. The background field in that case is the metric, $g^{\mu\nu}$. " $f^{abc}\bar{A}^c_\mu \sim \Gamma^{ab}_\mu$ "

The quadratic form $F^2$ then becomes:

redundant since $\bar{F}_{\mu\nu}$ already antisymmetric $\Rightarrow 4\bar{F}^a_{\mu\nu}(\bar{D}^\mu A^\nu)^a$ (integrate by parts)

$$-\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} \longrightarrow -\frac{1}{4}\Big(\bar{F}^a_{\mu\nu}\bar{F}^{\mu\nu a} + 2\bar{F}^a_{\mu\nu}(\bar{D}^\mu A^{\nu a} - \bar{D}^\nu A^{\mu a}) - \underline{2g f^{abc}\bar{F}^a_{\mu\nu}A^{\mu b}A^{\nu c}}$$

$$+ \underbrace{(\bar{D}^\mu A^{\nu a} - \bar{D}^\nu A^{\mu a})^2} - 2g f^{abc}(\bar{D}_\mu A^a_\nu - \bar{D}_\nu A^a_\mu)A^{\mu b}A^{\nu c}$$

$$+ g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{\mu d} A^{\nu e}\Big)$$

Clean up terms quadratic in quantum fields, $A^a_\mu$ :

$$\text{Quadratic Part} = \frac{-1}{4}\left[(\bar{D}^\mu A^{\nu a} - \bar{D}^\nu A^{\mu a})^2 - 2g f^{abc}\bar{F}^a_{\mu\nu} A^{\mu b} A^{\nu c}\right]$$

$$= \underbrace{-\frac{1}{2}(\bar{D}^\mu A^\nu)^a(\bar{D}_\mu A_\nu)^a}_{\text{Two square terms}} + \underbrace{\frac{1}{2}(\bar{D}^\mu A^\nu)^a(\bar{D}_\nu A_\mu)^a}_{\text{Two cross terms}} + \frac{1}{2} g f^{abc}\bar{F}^a_{\mu\nu} A^{\mu b} A^{\nu c}$$

Integrate by parts in first two terms:

$$(\bar{D}^\mu A^\nu)(\bar{D}_\mu A_\nu)^a = (\partial^\mu + g f^{abc}\bar{A}^{\mu c})A^{\nu b}(\bar{D}_\mu A_\nu)^a$$

$$\xrightarrow{\text{IBP}} A^{\nu b}(-\partial^\mu + g f^{abc}\bar{A}^{\mu c})(\bar{D}_\mu A_\nu)^a$$

$$= A^{\nu b}(-\partial^\mu - g f^{bac}\bar{A}^{\mu c})(\bar{D}_\mu A_\nu)^a$$

Notice how second term acquires a minus sign since $f^{abc} = -f^{bac}$.

$$= -A^{\nu b}(\bar{D}^\mu)^{ba}(\bar{D}_\mu A_\nu)^a$$

$$= -A^a_\mu(\bar{D}^2)^{ab} g^{\mu\nu} A^b_\nu \longleftarrow$$

Verified: an integration by parts for covariant derivatives works the same as for ordinary derivatives.

Similarly $(\bar{D}^\mu A^\nu)^a(\bar{D}_\nu A_\mu)^a = -A^a_\mu(\bar{D}^\nu\bar{D}^\mu)^{ab}A^b_\nu$.

So, finally,

$$= \frac{1}{2}A^a_\mu\left[(\bar{D}^2)^{ab}g^{\mu\nu} - (\bar{D}^\nu\bar{D}^\mu)^{ab}\right]A^b_\nu + \frac{1}{2} g \overset{\text{cyclic}}{f^{abc}}\bar{F}^a_{\mu\nu} A^{\mu b} A^{\nu c}$$

rename: $a\to c$, $b\to a$, $c\to b$

$$+\frac{1}{2} g f^{abc}\bar{F}^{\mu\nu c} A^a_\mu A^b_\nu$$

$$\mathcal{L}_{\text{Quad}} = \frac{1}{2}A^a_\mu\left[(\bar{D}^2)^{ab}g^{\mu\nu} - (\bar{D}^\nu\bar{D}^\mu)^{ab} + g f^{abc}\bar{F}^{\mu\nu c}\right]A^b_\nu$$

$$= \frac{1}{2}A^a_\mu\left[(\bar{D}^2)^{ab}g^{\mu\nu} + \underbrace{[\bar{D}^\mu,\bar{D}^\nu]^{ab}}_{g f^{abc}\bar{F}^{\mu\nu c}} - (\bar{D}^\mu\bar{D}^\nu)^{ab} + g f^{abc}\bar{F}^{\mu\nu c}\right]A^b_\nu$$

$$= \frac{1}{2}A^a_\mu\left[(\bar{D}^2)^{ab}g^{\mu\nu} - (\bar{D}^\mu\bar{D}^\nu)^{ab} + 2g f^{abc}\bar{F}^{\mu\nu c}\right]A^b_\nu$$

When quantizing the system, $\mathcal{L}_{GF} = \frac{-1}{2\xi}(\mathcal{F}^a)^2$ is added.

Choose $\mathcal{F}^a = \bar{D}^\mu A_\mu^a$, (BACKGROUND FIELD GAUGE) so that

$$\mathcal{L}_{GF} = \frac{-1}{2\xi}(\bar{D}^\mu A_\mu^a)^2 = \frac{-1}{2\xi}(\bar{D}^\mu A_\mu)^a(\bar{D}^\nu A_\nu)^a$$

$$\xrightarrow{\text{Int. by parts}} \frac{+1}{2\xi} A_\mu^a (\bar{D}^\mu \bar{D}^\nu)^{ab} A_\nu^b$$

Now, work on the compensating Fadeev-Popov determinant $\det\left(\frac{\delta\mathcal{F}(A')}{\delta\alpha}\right)$

$$\mathcal{F}^a(A') = \bar{D}^\mu A_\mu'^a$$

Note: $A_\mu^a \xrightarrow{\text{Gauge}} A_\mu^a - \frac{1}{g}(\bar{D}_\mu\alpha)^a - f^{abc}A_\mu^b\alpha^c$

$$= \bar{D}^\mu\left(A_\mu - \frac{1}{g}(\bar{D}_\mu\alpha)^a - f^{abc}A_\mu^b\alpha^c\right)$$

$$= (\partial^\mu A_\mu^a + g f^{abc}\bar{A}^{\mu c}A_\mu^b) - \frac{1}{g}(\partial^\mu\delta^{ab} + g f^{abc}\bar{A}^{\mu c})(\partial_\mu\delta^{bd} + g f^{bde}\bar{A}_\mu^e)\alpha^d$$
$$- f^{abc}(\partial^\mu\delta^{bd} + g f^{bde}\bar{A}^{\mu e})A_\mu^d\alpha^c$$

$$= (\partial^\mu A_\mu^a + g f^{abc}\bar{A}^{\mu c}A_\mu^b) - \frac{1}{g}\partial^\mu\partial_\mu\alpha^a - f^{ade}\left((\partial^\mu\bar{A}_\mu^e)\alpha^d + \underline{\bar{A}_\mu^e\partial^\mu\alpha^d}\right)$$
$$\underline{-f^{abc}\bar{A}^{\mu c}\partial_\mu\alpha^b} - g f^{abc}f^{bde}\bar{A}^{\mu c}\bar{A}_\mu^e\alpha^d$$
$$-f^{abc}\left((\partial^\mu A_\mu^b)\alpha^c + A_\mu^b(\partial^\mu\alpha^c) + g f^{bde}\bar{A}^{\mu e}A_\mu^d\alpha^c\right)$$

underlined terms add.

$$\frac{\delta\mathcal{F}'^a(x)}{\delta\alpha^b(y)} = \frac{\partial\mathcal{F}'^a}{\partial\alpha^b}\delta^{(4)}(x-y) + \frac{\partial\mathcal{F}'^a}{\partial(\partial^\mu\alpha^b)}\partial^\mu\delta^{(4)}(x-y) + \frac{\partial\mathcal{F}'^a}{\partial(\partial^\mu\partial^\nu\alpha^b)}\partial^\mu\partial^\nu\delta^{(4)}(x-y)$$

So,

$$\frac{\delta\mathcal{F}'^a(x)}{\delta\alpha^b(y)} = \Big(-f^{abc}\partial^\mu\bar{A}_\mu^c - g^2 f^{adc}f^{bde}\bar{A}^{\mu c}\bar{A}_\mu^e - f^{acb}\partial^\mu A_\mu^c$$
$$- g f^{acb}f^{cde}\bar{A}^{\mu e}A_\mu^d\Big)\delta^{(4)}(x-y)$$
$$+\left(-2f^{abc}\bar{A}_\mu^c - f^{abc}A_\mu^c\right)\partial^\mu\left(\delta^{(4)}(x-y)\,\cdot\right) - \frac{1}{g}g_{\mu\nu}\delta^{ab}\partial^\mu\partial^\nu\left(\delta^{(4)}(x-y)\,\cdot\right)$$

$$= \Big[ -\frac{1}{g}\delta^{ab}\partial^2 - 2f^{abc}\bar{A}^c_\mu \partial^\mu - f^{abc}\partial^\mu A^c_\mu - g f^{ade} f^{bde} \bar{A}^c \bar{A}^e$$

$$- f^{abc}\left(\partial^\mu A^c_\mu + A^c_\mu \partial^\mu + g f^{cde}\bar{A}^e A^d_\mu\right)\Big] \delta^{(4)}(x-y)$$

First line $\equiv \frac{-1}{g}(\bar{D}_\mu \bar{D}^\mu)^{ab}$ and Second line $\equiv -f^{abc}(\bar{D}_\mu A^\mu)^c$.

So,

$$\frac{\delta \mathcal{F}^a(x)}{\delta \alpha^b(y)} = \Big[ -\frac{1}{g}(\bar{D}_\mu \bar{D}^\mu)^{ab} - f^{abc}(\bar{D}_\mu A^\mu)^c \Big] \delta^{(4)}(x-y)$$

Hence, the Ghost lagrangian is, (after rescaling $\eta \to \sqrt{g}\,\eta$)

$$\mathcal{L}_{Ghost} = \eta^{\dagger a}\Big[ -(\bar{D}_\mu \bar{D}^\mu)^{ab} - g f^{abc}(\bar{D}_\mu A^\mu)^c \Big]\eta^b$$

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So, organizing the full Lagrangian in powers of quantum fields we have:

$$\mathcal{L} = \mathcal{L}^{[0]+lin} + \mathcal{L}^{Quad} + \mathcal{L}^{int}$$

$j$ chosen to remove linear terms.

$$\mathcal{L}^{[0]+lin} = -\frac{1}{4}\bar{F}^a_{\mu\nu}\bar{F}^{\mu\nu a} + j^a_\nu \bar{A}^{a\nu} + \left[(\bar{D}^\mu \bar{F}_{\mu\nu})^a + j^a_\nu\right] A^{\nu a}$$

$$\mathcal{L}^{Quad} = \frac{1}{2}\phi_i\left(-(\bar{D}^2)_{ij} - M^2_{ij}\right)\phi_j + \bar{\eta}^b\left[-(\bar{D}^2)^{ab}\right]\eta^b$$

$$+ \frac{1}{2}A^a_\mu\left[(\bar{D}^2)^{ab} g^{\mu\nu} - \left(1-\frac{1}{\xi}\right)(\bar{D}^\mu \bar{D}^\nu)^{ab} + 2g f^{abc}\bar{F}^{\mu\nu c}\right] A^a_\nu$$

Fluctuation operator for spin-1 fields

$$\mathcal{L}^{int} = \frac{1}{2} g f^{abc}(\bar{D}_\mu A_\nu - \bar{D}_\nu A_\mu)^a A^{\mu b} A^{\nu c} - \frac{g^2}{4} f^{abc} f^{ade} A^a_\mu A^c_\nu A^{\mu d} A^{\nu e} - V(\phi)$$

Pure gauge interaction terms:

$$\mathcal{L}_{int} = \frac{1}{2} g f^{abc} (\bar{D}_\mu A_\nu^a - \underset{\substack{\uparrow \\ \text{redundant}}}{\bar{D}_\nu A_\mu^a}) \underbrace{A^{\mu b} A^{\nu c}}_{\text{antisymmetric}} - \frac{g^2}{4} \overbrace{f^{abc}}^{\text{cyclic}} f^{ade} A_\mu^a A_\nu^c A^{\mu d} A^{\nu e}$$

rename: $b \to a \to c \to b$

$$= g f^{abc} (\bar{D}_\mu A_\nu^a) A^{\mu b} A^{\nu c} - \frac{g^2}{4} f^{abc} f^{cde} A_\mu^a A_\nu^b A^{\mu d} A^{\nu e}$$

$$= g f^{abc} (\partial_\mu A_\nu^a) A^{\mu b} A^{\nu c} + g^2 \overbrace{f^{abc}}^{\text{cyclic}} f^{ade} \bar{A}_\mu^d A_\nu^e A^{\mu b} A^{\nu c} - \frac{g^2}{4} f^{abc} f^{cde} A_\mu^a A_\nu^b A^{\mu d} A^{\nu e}$$

rename: $b \to a \to c \to b$

$$= g f^{abc} (\partial_\mu A_\nu^a) A^{\mu b} A^{\nu c} + \underbrace{g^2 f^{abc} f^{cde} A_\mu^a A_\nu^b \bar{A}^{\mu d} A^{\nu e}}_{\text{(New in BG field gauge)}} - \frac{g^2}{4} f^{abc} f^{cde} A_\mu^a A_\nu^b A^{\mu d} A^{\nu e}$$

Ghost interactions:

$$\mathcal{L}_{ghost} = - g f^{abc} \eta^{\dagger a} (\bar{D}_\mu A^\mu)^c \eta^b$$

$$= - g f^{abc} \eta^{\dagger a} \partial_\mu A^{\mu c} \eta^b - g^2 f^{abc} f^{cde} \eta^{\dagger a} \bar{A}_\mu^e A^{\mu d} \eta^b .$$

← (New in BG field gauge)

## Identities of the spin-1 fluctuation operator (no gauge fixing)

Hungry derivative: acts all the way to the right.

[non-abelian analog of the abelian $\partial_\mu(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)\phi = 0$]

$$D_\mu^{ab}\left((D^2)^{bc} g^{\mu\nu} - (D^\mu D^\nu)^{bc} + 2(F^{\mu\nu})^{bc}\right)\phi = (D_\mu F^{\mu\nu})^{ac}\phi$$

where $(F^{\mu\nu})^{bc} = g f^{abc} F^{\mu\nu\, a}$

**Proof:** Everything in color-matrix multiplication order – drop color indices.

Useful to note: $[D^\mu, D^\nu] = F^{\mu\nu} \Rightarrow D^2 D^\nu - D_\mu D^\nu D^\mu = D_\mu F^{\mu\nu}$

$$\text{LHS} = (\underbrace{D^\nu D^2}_{\text{commute}} - D^2 D^\nu + 2 D_\mu F^{\mu\nu})\phi$$

$$= (D^\mu D^\nu D_\mu + [D^\nu, D^\mu] D_\mu - D^2 D^\nu + 2 D_\mu F^{\mu\nu})\phi$$

$$= ([D^\nu, D^\mu] D_\mu - \cancel{D_\mu F^{\mu\nu}} + \cancel{2} D_\mu F^{\mu\nu})\phi$$

$$= (-F^{\mu\nu} D_\mu + D_\mu F^{\mu\nu})\phi$$

apply product rule

$$= \cancel{-F^{\mu\nu} D_\mu \phi} + (D_\mu F^{\mu\nu})\phi + \cancel{F^{\mu\nu} D_\mu \phi}$$

$$= (D_\mu F^{\mu\nu})\phi \quad ✓$$

Use similar manipulations to prove:

$$\left((D^2)^{ab} g^{\mu\nu} - (D^\mu D^\nu)^{ab} + 2(F^{\mu\nu})^{ab}\right) D_\mu^{bc}\phi = (D_\mu F^{\mu\nu})^{ac}\phi$$