see Parthasarathy. J. Phys. A. Gen 21 (1988) 4593–4607 for discussion of non-linear gauges.

Gauge Fixing (General Class of Linear Gauges)

Start with pure Yang-Mills theory — take SU(N) as the symmetry group.

$$\mathcal{L}_{YM} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a}$$

$$= \frac{1}{2} A^a_\mu (\partial^2 g^{\mu\nu} - \partial^\mu \partial^\nu) A^a_\nu \quad + \text{“} g A^3 + g^2 A^4 \text{”}$$

Impose gauge fixing condition $\mathcal{F}^a = f^\mu A^a_\mu = 0$ $\qquad [f^\mu] = [\text{Energy}]^1$

$\uparrow$ arbitrary gauge fixing vector

$f^\mu \sim p^\mu \equiv$ Lorentz gauges

$\sim \vec{p} =$ Coulomb gauge

$f^\mu =$ fixed $\Rightarrow$ Axial gauges.

Following the Fadeev-Popov gauge fixing procedure, the gauge fixing condition is implemented by adding to the Lagrangian.

$$\mathcal{L}_{YM+GF} = \frac{1}{2} A^a_\mu (\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu) A^a_\nu - \frac{1}{2\xi}(f^\mu A^a_\mu)^2 \quad + \text{“} gA^3 + g^2 A^4 \text{”}$$

$\longrightarrow$ move to momentum space

$$= \frac{1}{2} \tilde{A}^a_\mu \left(-p^2 g^{\mu\nu} + p^\mu p^\nu - \frac{1}{\xi} f^\mu f^\nu\right) \tilde{A}^a_\nu \quad + \text{“} gA^3 + g^2 A^4 \text{”}$$

Decompose gauge-fixing vector f^μ into a part parallel and orthogonal to p^μ:

$$f^\mu = \underbrace{\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) f_\nu}_{\equiv n^\mu} + \frac{p.f}{p^2} p^\mu = n^\mu + \frac{p.f}{p^2} p^\mu$$

Note: $n.p = 0$, $\quad n^2 = f^2 - \dfrac{(p.f)^2}{p^2}$

f^μ has units $[E]^1$.

Then,

$$f^\mu f^\nu = n^\mu n^\nu + \frac{p.f}{p^2}\left(n^\mu p^\nu + n^\nu p^\mu\right) + \frac{(p.f)^2}{p^2}\frac{p^\mu p^\nu}{p^2}$$

So, the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2}\tilde{A}^a_\mu\left[-p^2 g^{\mu\nu} + \underbrace{p^\mu p^\nu}_{\equiv\, p^2\frac{p^\mu p^\nu}{p^2}} - \underbrace{\frac{1}{\xi}n^\mu n^\nu}_{\equiv\, \frac{1}{\xi}n^2\frac{n^\mu n^\nu}{n^2}} - \frac{1}{\xi}\frac{p.f}{p^2}\left(n^\mu p^\nu + n^\nu p^\mu\right) - \frac{1}{\xi}\frac{(p.f)^2}{p^2}\frac{p^\mu p^\nu}{p^2}\right]\tilde{A}^a_\nu$$

added/subtracted

$$= \frac{1}{2}\tilde{A}^a_\mu\left[-p^2\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \frac{n^\mu n^\nu}{n^2}\right) - p^2\frac{n^\mu n^\nu}{n^2} - \frac{1}{\xi}n^2\frac{n^\mu n^\nu}{n^2}\right.$$
$$\left. - \frac{1}{\xi}\frac{p.f}{p^2}\left(n^\mu p^\nu + n^\nu p^\mu\right) - \frac{1}{\xi}\frac{(p.f)^2}{p^2}\frac{p^\mu p^\nu}{p^2}\right]\tilde{A}^a_\nu$$

$$= \frac{1}{2}A^a_\mu\left[-p^2\overbrace{\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \frac{n^\mu n^\nu}{n^2}\right)}^{\equiv A^{\mu\nu}} - \left(p^2 + \frac{1}{\xi}n^2\right)\overbrace{\frac{n^\mu n^\nu}{n^2}}^{\equiv B^{\mu\nu}}\right.$$
$$\left. - \frac{1}{\xi}\frac{p.f}{p^2}\underbrace{\left(n^\mu p^\nu + n^\nu p^\mu\right)}_{\equiv C^{\mu\nu}} - \frac{1}{\xi}\frac{(p.f)^2}{p^2}\underbrace{\frac{p^\mu p^\nu}{p^2}}_{\equiv E^{\mu\nu}}\right]\tilde{A}^a_\nu$$

$$= \frac{1}{2}A^a_\mu\left[-p^2 A^{\mu\nu} - \left(p^2 + \frac{1}{\xi}n^2\right)B^{\mu\nu} - \frac{1}{\xi}\frac{p.f}{p^2}C^{\mu\nu} - \frac{1}{\xi}\frac{(p.f)^2}{p^2}E^{\mu\nu}\right]\tilde{A}^a_\nu$$

Let $a = -p^2$, $\quad b = -\left(p^2 + \frac{1}{\xi}n^2\right)$

$c = -\frac{1}{\xi}\frac{p.f}{p^2}$, $\quad e = -\frac{1}{\xi}\frac{(p.f)^2}{p^2}$

The propagator $\equiv \tilde{D}^F_{\mu\nu}$ is obtained by $\tilde{\mathcal{O}}^{\mu\nu}\left(-i\tilde{D}^F_{\nu\rho}\right) = \delta^\mu{}_\rho$.

i.e finding the inverse of $\tilde{\mathcal{O}}^{\mu\nu}$

Ansatz:

$$-i\tilde{D}^F_{\nu\rho} = a' A_{\nu\rho} + b' B_{\nu\rho} + c' C_{\nu\rho} + e' E_{\nu\rho}\,,$$

a', b', c', e' to be solved for.

Then, we must have $\tilde{\mathcal{O}}^{\mu\nu}(-i\tilde{D}^F_{\nu\rho}) = \delta^\mu{}_\rho$, or

$$\left(a A^{\mu\nu} + b B^{\mu\nu} + c C^{\mu\nu} + e E^{\mu\nu}\right)\left(a' A_{\nu\rho} + b' B_{\nu\rho} + c' C_{\nu\rho} + e' E_{\nu\rho}\right) = \delta^\mu{}_\rho .$$

IDENTITIES — see next 2 pages

$$A^{\mu\nu}A_{\nu\rho} = A^\mu{}_\rho \qquad B^{\mu\nu}B_{\nu\rho} = B^\mu{}_\rho \qquad C^{\mu\nu}C_{\nu\rho} = p^2 n^2 (B^\mu{}_\rho + E^\mu{}_\rho)$$
$$A^{\mu\nu}B_{\nu\rho} = 0 \qquad B^{\mu\nu}C_{\nu\rho} = n^\mu p_\rho \qquad C^{\mu\nu}E_{\nu\rho} = n^\mu p_\rho$$
$$A^{\mu\nu}C_{\nu\rho} = 0 \qquad B^{\mu\nu}E_{\nu\rho} = 0 \qquad E^{\mu\nu}E_{\nu\rho} = E^\mu{}_\rho .$$
$$A^{\mu\nu}E_{\nu\rho} = 0$$

(all identities hold even when $C_{\nu\rho} \to n_\nu p_\rho$)

Since $A^{\mu\nu}$ is orthogonal to all other tensors, we can take $a' = \frac{1}{a}$.

$$\Rightarrow A^\mu{}_\rho + \left(bb' B^\mu{}_\rho + bc' n^\mu p_\rho\right) + \left(cb' p^\mu n_\rho + cc' p^2 n^2 (B^\mu{}_\rho + E^\mu{}_\rho) + ce' n^\mu p_\rho\right) + \left(ec' p^\mu n_\rho + ee' E^\mu{}_\rho\right) = \delta^\mu{}_\rho$$

NOTICE that $A^\mu{}_\rho + B^\mu{}_\rho + E^\mu{}_\rho = \delta^\mu{}_\rho$. The above can be suggestively reorganized:

$$\Rightarrow A^\mu{}_\rho + (bb' + cc' p^2 n^2) B^\mu{}_\rho + (cc' p^2 n^2 + ee') E^\mu{}_\rho + (bc' + ce') n^\mu p_\rho + (cb' + ec') p^\mu n_\rho = \delta^\mu{}_\rho .$$

Hence, we would like 1^{st} line to equal $A+B+E = \delta$ to equal 1 and 2^{nd} line to equal 0.

$$\left.\begin{aligned} bb' + cc' p^2 n^2 &= 1 \\ ee' + cc' p^2 n^2 &= 1 \\ bc' + ce' &= 0 \\ cb' + ec' &= 0 \end{aligned}\right\} \Rightarrow \quad b' = \frac{e}{be - c^2 p^2 n^2}, \quad c' = \frac{-c}{be - c^2 p^2 n^2}, \quad e' = \frac{b}{be - c^2 p^2 n^2}$$

Derivation of IDENTIES

Abbreviate $A^{\mu\nu} \equiv g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \frac{n^\mu n^\nu}{n^2}$ unitless

$B^{\mu\nu} \equiv \frac{n^\mu n^\nu}{n^2}$ unitless

$C^{\mu\nu} \equiv n^\mu p^\nu + n^\nu p^\mu$

$E^{\mu\nu} \equiv \frac{p^\mu p^\nu}{p^2}$ unitless, where

$n^\mu = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) f_\nu$:

$n^\mu f_\mu = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) f_\nu f_\mu = f^2 - \frac{(f.p)^2}{p^2}$

$n^\mu p_\mu = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) f_\nu p_\mu = f.p - f.p =$

$n^\mu n_\mu = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) f_\nu \left(\delta_\mu{}^\rho - \frac{p_\mu p^\rho}{p^2}\right) f_\rho$
$= f^2 - \frac{2(p.f)^2}{p^2} + \frac{(p.f)^2}{p^2} = f^2 - \frac{(p.f)^2}{p^2}$

$$A^{\mu\nu} A_{\nu\rho} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \frac{n^\mu n^\nu}{n^2}\right)\left(g_{\nu\rho} - \frac{p_\nu p_\rho}{p^2} - \frac{n_\nu n_\rho}{n^2}\right)$$

$$= \left[\delta^\mu{}_\rho - \frac{p^\mu p_\rho}{p^2} - \frac{n^\mu n_\rho}{n^2} - \cancel{\frac{p^\mu p_\rho}{p^2}} + \cancel{\frac{p^\mu p_\rho}{p^2}} + 0 \right.$$
$$\left. - \cancel{\frac{n^\mu n_\rho}{n^2}} + 0 \qquad + \cancel{\frac{n^\mu n_\rho}{n^2}}\right]$$

$$= \delta^\mu{}_\rho - \frac{p^\mu p_\rho}{p^2} - \frac{n^\mu n_\rho}{n^2} = A^\mu{}_\rho$$

$$A^{\mu\nu} B_{\nu\rho} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \frac{n^\mu n^\nu}{n^2}\right)\frac{n_\nu n_\rho}{n^2}$$

$$= \frac{n^\mu n_\rho}{n^2} - 0 - \frac{n^\mu n_\rho}{n^2} = 0.$$

$$A^{\mu\nu} C_{\nu\rho} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \frac{n^\mu n^\nu}{n^2}\right)(n_\nu p_\rho + n_\rho p_\nu) =$$

$$= \cancel{n^\mu p_\rho} - 0 - \cancel{n^\mu p_\rho} + (\mu \leftrightarrow \rho) = 0$$

$$A^{\mu\nu} E_{\nu\rho} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} - \frac{n^\mu n^\nu}{n^2}\right)\frac{p_\nu p_\rho}{p^2}$$

$$= \frac{p^\mu p_\rho}{p^2} - \frac{p^\mu p_\rho}{p^2} - 0 = 0.$$

Hence, the inverse is :

$$-i\tilde{D}^{\mu\nu} = \frac{1}{a}A^{\mu\nu} + \frac{1}{be - c^2p^2n^2}\left(eB^{\mu\nu} - cC^{\mu\nu} + bE^{\mu\nu}\right)$$

simplify: $\quad be - c^2p^2n^2 = \frac{1}{\xi}\frac{(p\cdot f)^2}{p^2}\left(p^2 + \frac{n^2}{\xi}\right) - \frac{1}{\xi^2}\frac{(p\cdot f)^2}{(p^2)^2}p^2n^2$

$$= \frac{1}{\xi^2}\frac{(p\cdot f)^2}{p^2}\left[\xi p^2 + \cancel{n^2} - \cancel{\frac{1}{p^2}p^2n^2}\right]$$

$$= \frac{1}{\xi^{\cancel{2}}}\frac{(p\cdot f)^2}{\cancel{p^2}}\cancel{\xi}\cancel{p^2} = \frac{(p\cdot f)^2}{\xi}$$

So,

$$-i\tilde{D}^{\mu\nu} = -\frac{1}{p^2}A^{\mu\nu} + \frac{\xi}{(p\cdot f)^2}\left[-\frac{1}{\xi}\frac{(p\cdot f)^2}{p^2}B^{\mu\nu} + \frac{1}{\xi}\frac{p\cdot f}{p^2}C^{\mu\nu} - \left(p^2 + \frac{n^2}{\xi}\right)E^{\mu\nu}\right]$$

$$= \underbrace{-\frac{1}{p^2}A^{\mu\nu} - \frac{1}{p^2}B^{\mu\nu}} + \frac{C^{\mu\nu}}{p^2(p\cdot f)} - \frac{1}{(p\cdot f)^2}\left(\xi p^2 + n^2\right)E^{\mu\nu}$$

(propagator) $$\boxed{\tilde{D}^{\mu\nu} = -\frac{i}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + \frac{iC^{\mu\nu}}{p^2(p\cdot f)} - \frac{i}{(p\cdot f)^2}\left(\xi p^2 + n^2\right)\frac{p^\mu p^\nu}{p^2}}$$

where $C^{\mu\nu} = n^\mu p^\nu + n^\nu p^\mu$,

and $n^\mu = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)f_\nu$

Special case: class of covariant gauges :

$$f^\mu = -ip^\mu \to p^\mu$$

(integration by parts)

$$\Rightarrow \quad n^\mu = 0, \quad C^{\mu\nu} = 0$$

Hence, $\tilde{D}^{\mu\nu} = \frac{-i}{p^2}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) - \frac{i}{(p^2)^2}\left(\xi p^2\right)\frac{p^\mu p^\nu}{p^2}$

$$= \frac{-i}{p^2}\left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$$