Conservation Laws from Functional Methods

Context: Complex scalar field theory

$$\mathcal{L} = (\partial_\mu \phi)^* \partial^\mu \phi - m^2 \phi^* \phi$$

— invariant under $\phi \to e^{i\alpha}\phi$.

Allow $\alpha \to \alpha(x)$ to be a space-time dependent parameter. Implement such an infinitessimal transformation as a change of variables in the functional integral.

$$\phi(x) \longrightarrow \phi(x) + i\alpha(x)\phi(x)$$

$$\int \mathcal{D}\phi \;\; \phi(x_1)\phi^*(x_2) \; e^{i\int d^4x\, \mathcal{L}[\phi]} = \int \mathcal{D}\phi' \;\; \phi'(x_1)\,\phi'^*(x_2)\; e^{i\int d^4x\,\mathcal{L}[\phi']}$$

Expand to first order in $\alpha(x)$ — leading terms cancel. $\mathcal{D}\phi' = \mathcal{D}\phi$ (Jac. det $= 1$)

$$0 = \int \mathcal{D}\phi \Big\{ \big(i\alpha(x_1)\phi(x_1)\big)\phi^*(x_2) - i\phi(x_1)\big(\alpha(x_2)\phi^*(x_2)\big) + \phi(x_1)\phi^*(x_2)\, i\int d^4x\; \partial_\mu\alpha\,(-i\phi^* \overleftrightarrow{\partial}^\mu \phi) \Big\}$$

$$+ \phi(x_1)\phi^*(x_2)\, i\int d^4x \Big[\partial_\mu\alpha\; i(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi)\Big]\Big\} e^{i\int d^4x\,\mathcal{L}[\phi]}$$

integrate by parts.

$$0 = \int \mathcal{D}\phi \int d^4x\; \alpha(x) \Big\{ i\phi(x_1)\,\delta^4(x-x_1)\,\phi^*(x_2) - i\phi(x_1)\phi^*(x_2)\,\delta^4(x-x_2) + \phi(x_1)\phi^*(x_2)\, i\,\partial_\mu j^\mu(x)\Big\} e^{i\int d^4x\,\mathcal{L}}$$

upon dividing by Z

$$0 = i\,\langle 0|\, T\big(\phi(x_1)\,\delta^4(x-x_1)\,\phi^*(x_2)\big)|0\rangle - i\,\langle 0|\, T\big(\phi(x_1)\phi^*(x_2)\,\delta^4(x-x_2)\big)|0\rangle$$

$$+ \langle 0|\, T\big(\phi(x_1)\phi^*(x_2)\; i\,\partial_\mu j^\mu(x)\big)|0\rangle$$

or

$$\langle 0|\, T\big(\phi(x_1)\phi^*(x_2)\,\partial_\mu j^\mu(x)\big)|0\rangle = \underbrace{\langle 0|\, T\big(\phi(x_1)\,\delta^{(4)}(x-x_1)\,\phi^*(x_2)\big)|0\rangle - \langle 0|\, T\big(\phi(x_1)\,\phi^*(x_2)\,\delta^{(4)}(x-x_2)\big)|0\rangle}_{\text{contact terms}}.$$

Take Fourier transform:

Recall, $\tilde{G}^{[n]}(p_1, \ldots, p_n) = \int d^4x_1 \cdots d^4x_n \; e^{-ip_1\cdot x_1 \cdots - ip_n\cdot x_n} \, G^{[n]}(x_1, \ldots, x_n)$

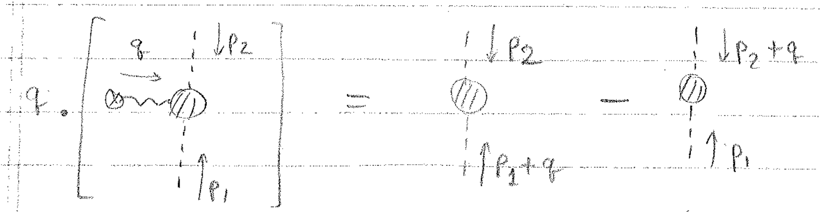
Apply $\int d^4x_1 \, d^4x_2 \, d^4x \; e^{-ip_1\cdot x_1 - ip_2\cdot x_2 - iq\cdot x}$

$$q_\mu \tilde{G}^{[3]}(p_1, p_2, q)^\mu = \int d^4x_1 \, d^4x_2 \; e^{-ip_2\cdot x_2 - i(p_1+q)\cdot x_1} \langle 0| T(\phi(x_1)\, \phi^*(x_2))|0\rangle$$

$$- \int d^4x_1 \, d^4x_2 \; e^{-ip_1\cdot x_1 - i(p_2+q)\cdot x_2} \langle 0| T(\phi(x_1)\, \phi^*(x_2))|0\rangle$$

$$\boxed{q_\mu \tilde{G}^{[3]}(p_1, p_2, q)^\mu = \tilde{G}^{[2]}(p_1+q, p_2) - \tilde{G}^{[2]}(p_1, p_2+q)}$$

This is the Ward identity for all Green's functions.



$\tilde{G}(-p_2) \quad - \quad \tilde{G}(p_1)$