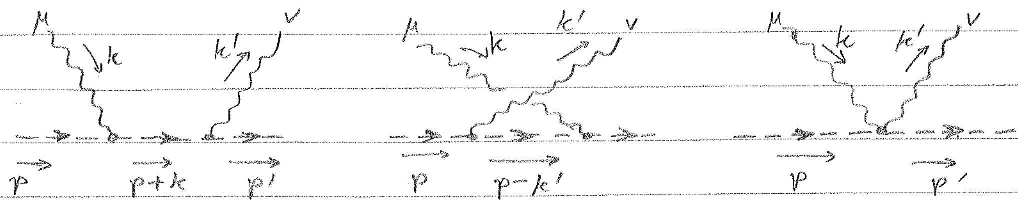
Ward identity for $\gamma\pi^+ \to \gamma\pi^+$ — Scalar QED

Pion Compton scattering:



kinematics: $p+k = p'+k'$
$k^2 = k'^2 = 0$
$p^2 = p'^2 = m^2$

3 diagrams

$$i\mathcal{M} = \Big\{ (-ie)(2p'+k')^\nu \frac{i}{(p+k)^2-m^2} (-ie)(p+k+p)^\mu + (-ie)(2p'-k)^\mu \frac{i}{(p-k')^2-m^2} (-ie)(p-k'+p)^\nu + 2ie^2 g^{\mu\nu} \Big\} \epsilon_\mu(k)\, \epsilon^*_\nu(k')$$

$$\mathcal{M} = -e^2 \left[\frac{(2p'+k')^\nu (2p+k)^\mu}{(p+k)^2 - m^2} + \frac{(2p'-k)^\mu (2p-k')^\nu}{(p-k')^2 - m^2} - 2g^{\mu\nu} \right] \epsilon_\mu(k)\, \epsilon^*_\nu(k')$$

SINGLE PHOTON IDENTITY

Replace $\epsilon_\mu(k) \to k_\mu$

No need to assume other photon is on-shell: $k'^2 = 0$, $k'\cdot\epsilon^* = 0$

$$k\cdot\mathcal{M} = -e^2 \left[\frac{(2p'+k')\cdot\epsilon^*(k')\ (2p+k)\cdot k}{(p+k)^2 - m^2} + \frac{(2p'-k)\cdot k\ (2p-k')\cdot\epsilon^*(k')}{(p-k')^2 - m^2} - 2k'\cdot\epsilon^* \right]$$

on shell

$$= -e^2 \left[\frac{(2p'+k')\cdot\epsilon^*(k')\ 2p\cdot k}{(p+k)^2 - m^2} + \frac{2p'\cdot k\ (2p-k')\cdot\epsilon^*(k')}{(p-k')^2 - m^2} - 2k'\cdot\epsilon^* \right]$$

Write: $2p\cdot k = \left[(p+k)^2 - m^2\right] - \left[(\underbrace{p-k'}_{p'-k})^2 - m^2\right] - 2p'\cdot k$

check:
$$= (\cancel{p^2} + 2p\cdot k + \underbrace{k^2}_{\to 0} - \cancel{m^2}) - (\cancel{p'^2} - 2p'\cdot k + \underbrace{k^2}_{\to 0} - \cancel{m^2}) - 2p'\cdot k$$
on-shell on-shell

$$= 2p\cdot k + \cancel{2p'\cdot k} - \cancel{2p'\cdot k} = 2p\cdot k \checkmark$$

(this is a difference of two propagators, and an extra part.

also $\quad 2p'\cdot k = [(p+k)^2 - m^2] - [(p-k')^2 - m^2] - 2p\cdot k$

(obtainable by solving for $2p'\cdot k$ in previous relation.)

$$k\cdot M = -e^2\left[\frac{(2p'+k')\cdot\epsilon^*\{[(p+k)^2-m^2] - [(p-k')^2-m^2] - 2p'\cdot k\}}{(p+k)^2 - m^2 \;\text{cancel}}\right.$$

$$\left. + \frac{\{[(p+k)^2-m^2] - [(p-k')^2-m^2] - 2p\cdot k\}(2p-k')\cdot\epsilon^*(k')}{(p-k')^2 - m^2 \;\text{cancel.}} - 2k'\cdot\epsilon^*\right]$$

$$k\cdot M = -e^2\left[\frac{(2p'+k')\cdot\epsilon^*\{-[(p-k')^2-m^2] - 2p'\cdot k\}}{(p+k)^2 - m^2} + (2p'+k')\cdot\epsilon^*\right.$$

$$\left. + \frac{\{[(p+k)^2-m^2] - 2p\cdot k\}(2p-k')\cdot\epsilon^*(k')}{(p-k')^2 - m^2} - (2p-k')\cdot\epsilon^* - 2k'\cdot\epsilon^*\right]$$

Quantities in $\{\ldots\}$ vanish:

① $\{-[(\underbrace{p-k'}_{p'-k})^2 - m^2] - 2p'\cdot k\} = \{-p'^2 + \cancel{2p'\cdot k} - \overset{\to 0}{k^2} + m^2 - \cancel{2p'\cdot k}\} = 0.$

on shell

② $\{[(p+k)^2 - m^2] - 2p\cdot k\} = \{p^2 + \cancel{2p\cdot k} + \overset{\to 0}{k^2} - m^2 - \cancel{2p\cdot k}\} = 0.$

on shell

$$k\cdot M = -e^2\left[(2p'+k')\cdot\epsilon^* - (2p-k')\cdot\epsilon^* - 2k\cdot\epsilon^*\right]$$

$$= -e^2\left[2p' + k' - 2p - k' - 2k\right]\cdot\epsilon^*(k')$$

$$= -e^2\left[2(\underbrace{p+p'-k-k'}_{=0})\right]\cdot\epsilon^*(k')$$

by conservation of momentum.

ϵ^* need not be transverse, nor $k^2 = 0$.

Ward id. holds even when 2^{nd} photon is off the mass shell.