Helmholtz-(Hodge) decomposition

A vector field can be split into transverse and longitudinal parts:

$$\underbrace{\vec{A}(\vec{x}) = \vec{A}_\perp(\vec{x})}_{\text{transverse}} + \underbrace{\vec{A}_\parallel(\vec{x})}_{\text{Longitudinal}}$$

IMPORTANT! $\vec{A}_\perp \cdot \vec{A}_\parallel \neq 0$, in general.

Transverse:

Projector: $(A_\perp)_i = (P_\perp)_{ij} A_j$

$$= \left[\delta_{ij} - \nabla_i \frac{1}{\nabla^2} \nabla_j\right] A_j$$

Property: $\vec{\nabla} \cdot \vec{A}_\perp(x) = 0$ for all $\vec{x}$.

Meaning

(defined in terms of what vector field doesn't look like).

Transverse vector field should produce no flux through small closed area around any point.

(constant fields are transverse)

Longitudinal:

Projector: $(A_\parallel)_i = (P_\parallel)_{ij} A_j$

$$= \left[\nabla_i \frac{1}{\nabla^2} \nabla_j\right] A_j$$

Property: $\vec{\nabla} \times \vec{A}_\parallel(x) = 0$ for all $\vec{x}$.

Longitudinal vector field should have no circulation around a small loop about any point.

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* *Note:*

$$\frac{1}{\nabla^2} f(x) := \int d^3\vec{x}\,' \frac{-1}{4\pi} \frac{1}{|\vec{x}-\vec{x}\,'|} f(\vec{x}\,')$$

stands for Green's function.