't Hooft vertex for SU(2)

Quick way to get 't Hooft vertex for higher reps. of SU(2). — Integrate anomaly eqn to determine charge in quanta.

Couple left-chiral fermion ψ_L (rep. j) to SU(2) gauge fields.

Consider the current carried by this fermion:

$$\partial_\mu J^\mu = \frac{g^2}{16\pi^2} \mathrm{Tr}\left[G \, \tfrac{1}{2}\{T_j^a, T_j^b\} \right] W^a_{\mu\nu} \tilde{W}^{\mu\nu\, a}$$

G = charge carried by fermion (prop. to id. matrix)

Recall: $\mathrm{Tr}[T_j^a, T_j^b] = \underbrace{\tfrac{1}{3} j(j+1)(2j+1)}_{\text{Dynkin index}} \delta^{ab}$

$$\partial_\mu J^\mu = \frac{g^2}{16\pi^2} G \, \tfrac{1}{\cancel{2}} \cancel{2} \cdot \tfrac{1}{3} j(j+1)(2j+1) \underbrace{\delta^{ab} W^a_{\mu\nu} \tilde{W}^{\mu\nu\, b}}_{\text{write in terms of matrix-valued fields.}}$$

$$2 \times \underbrace{\tfrac{1}{2}\delta^{ab}}_{\mathrm{Tr}[T^a_{1/2} T^b_{1/2}]} W^a \tilde{W}^b$$

$T_{1/2} \equiv$ fundamental representation

$$= \frac{g^2}{16\pi^2} G \, \tfrac{1}{3} j(j+1)(2j+1) \; 2\, \mathrm{Tr}[W_{\mu\nu} \tilde{W}^{\mu\nu}]$$

$$\frac{\partial \rho_G}{\partial t} + \vec{\nabla}\cdot \vec{J}_G = G \, \tfrac{2}{3} j(j+1)(2j+1) \; \frac{g^2}{16\pi^2} \mathrm{Tr}[W_{\mu\nu} \tilde{W}^{\mu\nu}]$$

Integrate over 4-volume:

$$\underbrace{\int_{-\infty}^{\infty} dt}_{\substack{\text{wick rotate} \\ t \to -i\tau}} \int d^3x \left(\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} \right) = G \, \tfrac{2}{3} j(j+1)(2j+1) \underbrace{\int dt}_{\substack{\text{wick rotate} \\ t \to -i\tau}} d^3x \, \mathrm{Tr}[W_{\mu\nu} \tilde{W}^{\mu\nu}]$$

First term: $\cancel{-i} \int d\tau \, \frac{\partial Q}{\partial \tau} = -i \Delta G$

$$\cancel{-i} \underbrace{\int d^4x_E \, \mathrm{Tr}[W_{\mu\nu} \tilde{W}^{\mu\nu}]}_{\nu}$$ (winding number of inst.)

Second term: $\longrightarrow 0$

$$\boxed{\frac{\Delta G}{G} = \Delta n_L = \tfrac{2}{3} j (j+1)(2j+1)} \equiv 2\times(\text{Dynkin index})$$

in agreement with Creutz. Ann Phys 323 (2008) p. 2349

explicitly,

$j = 1/2$: $\Delta n = 1$

$j = 1$: $\Delta n = 4$

$j = 3/2$: $\Delta n = 10$

$$\Rightarrow \quad \mathcal{L}_{\text{'t Hooft}} \sim (\psi_L)^{\Delta n_L}$$