Conversion from natural to SI units

space-time:

$$t_{SI} = \hbar t$$

$$\vec{x}_{SI} = \hbar c \vec{x}$$

$$\frac{\partial}{\partial t_{SI}} = \frac{1}{\hbar} \frac{\partial}{\partial t}$$

$$\vec{\nabla}_{SI} = \frac{1}{\hbar c} \vec{\nabla}$$

mechanical:

$$S_{SI} = \hbar S$$

$$L_{SI} = L$$

$$\mathcal{L}_{SI} = \frac{1}{(\hbar c)^3} \mathcal{L}$$

$$m_{SI} = \frac{m}{c^2}$$

$$p_{SI} = \frac{p}{c}$$

scattering variables

$$\sigma_{SI} = (\hbar c)^2 \sigma$$

Quantum electrodynamics:

$$\vec{E}_{SI} = \sqrt{\frac{\mu_0}{\hbar^3 c}} \vec{E}$$

$$\vec{B}_{SI} = \sqrt{\frac{\mu_0}{(\hbar c)^3}} \vec{B}$$

$$\phi_{SI} =$$

$$\vec{A}_{SI} = \sqrt{\frac{\mu_0}{\hbar c}} \vec{A}$$

$$|e|_{SI} = \sqrt{\frac{\hbar}{c \mu_0}}\, e = \frac{e}{\sqrt{\epsilon_0 \hbar c}} \Rightarrow \frac{|e|^2_{SI}}{4\pi\epsilon_0 \hbar c} = \frac{e^2}{4\pi}$$

$$\psi_{SI} = \frac{1}{(\hbar c)^{3/2}} \psi$$

Covariant quantities in SI units

SI units

$x^\mu = (ct; \vec{x})$ $\qquad [x^\mu] = m$

$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$ $\qquad [\partial_\mu] = m^{-1}$

$$F^{\mu\nu} = \left(\begin{array}{c|ccc} 0 & -E_x/c & -E_y/c & -E_z/c \\ \hline E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{array}\right)$$

$[F^{\mu\nu}] = \frac{kg}{C\,s} = \text{Tesla} = [\vec{B}]$

$\frac{kg\,m}{C\,s^2} = [\vec{E}]$

$A^\mu = \left(\frac{\phi}{c}, \vec{A}\right)$ $\qquad [A^\mu] = \frac{kg\,m}{C\,s} = \frac{\text{momentum}}{\text{charge}}.$

$J^\mu = (c\rho, \vec{J})$ $\qquad [J^\mu] = \frac{C}{m^2\,s}$

QED Lagrangian in SI

$$\mathcal{L}^{[SI]}_{QED} = \hbar c\, \bar{\psi}(i\not{D})\psi - mc^2\bar{\psi}\psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}$$

$D_\mu = \partial_\mu + i\frac{|e|}{\hbar}QA_\mu$

(notice: $\sqrt{\hbar}\,\bar{\psi}$ & $\sqrt{\hbar}\,\psi$)

$$= \hbar c\,\bar{\psi}(i\not{\partial})\psi - mc^2\bar{\psi}\psi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - c\,|e|\,Q\,A_\mu\bar{\psi}\gamma^\mu\psi$$

elementary charge $= 1.6\times10^{-19}\ C$

$[\mathcal{L}] = \frac{kg}{s^2\,m}$

$J^\mu = |e|Qc\,\bar{\psi}\gamma^\mu\psi$

$[\psi] = m^{-3/2}$ (like wavefunction)

$Q = -1$ for e^-/e^+

$\epsilon_0 = 8.85\times10^{-12}\ \frac{C^2 s^2}{kg\,m^3} = \frac{F}{m}$

$\mu_0 = 1.26\times10^{-6}\ \frac{kg\,m}{C^2} = \frac{H}{m}$

$c = \frac{1}{\sqrt{\epsilon_0\mu_0}} = 3.0\times10^8\ \frac{m}{s}$

Classical equations of motion — SI

$$\mathcal{L}_{QED}^{[SI]} = \hbar c\, \bar{\psi}\left(i\not{\partial} - \frac{mc^2}{\hbar c}\right)\psi - \frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - c|e|\, Q A_\mu \bar{\psi}\gamma^\mu \psi$$

$$\boxed{\bar{\psi}}: \quad \frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0$$

$$(\hbar c\, i\not{\partial} - mc^2)\psi - c|e|\, Q A_\mu \gamma^\mu \psi = 0.$$

$$\therefore \quad \left[\hbar c\, i\not{\partial} - c|e| Q A_\mu \gamma^\mu - mc^2\right]\psi = 0$$

$$\therefore \quad \left[i\hbar c\, \gamma^\mu \underbrace{\left(\partial_\mu + \frac{i|e|}{\hbar} Q A_\mu\right)}_{D_\mu} - mc^2\right]\psi = 0.$$

Note: If the Dirac equation is interpreted as a probabilistic wave equation, (like the Schrödinger eqn), then the Hamiltonian is obtained by multiplying at left by γ^0.

$$\underbrace{\gamma^0 i\hbar c\left[\gamma^0\right.}_{=1}\left(\underset{\frac{1}{c}\frac{\partial}{\partial t}}{\partial_0} + \frac{i|e|}{\hbar} Q \underset{\phi/c}{A_0}\right) + \left.\vec{\gamma}\cdot\left(\vec{\nabla} \underset{\text{metric}}{-} \frac{i|e|}{\hbar} Q\vec{A}\right)\right]\psi - \gamma^0 mc^2 \psi = 0$$

$$\left[i\hbar\frac{\partial}{\partial t} - c|e| Q\phi + i\hbar c\, \overbrace{\gamma^0\vec{\gamma}}^{\vec{\alpha}}\cdot\vec{\nabla} - c|e| Q\, \overbrace{\gamma^0\vec{\gamma}}^{\vec{\alpha}}\cdot\vec{A} - \overset{\beta}{\overset{\|}{\gamma^0}} mc^2\right]\psi$$

$$i\hbar\frac{\partial}{\partial t}\psi = \underbrace{\left[-i\hbar c\, \vec{\alpha}\cdot\vec{\nabla} + c|e| Q\, \vec{\alpha}\cdot\vec{A} + \gamma^0 mc^2 + |e| Q\phi\right]}_{\text{Dirac Hamiltonian}}\psi$$

SI

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu$$

① ②

$$\frac{\partial \mathcal{L}}{\partial A_\beta} - \partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = 0$$

① $$\frac{\partial \mathcal{L}}{\partial A_\beta} = -J^\beta$$

② $$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{\mu_0} F^{\alpha\beta}$$

$$\Rightarrow -\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = \frac{1}{\mu_0}\partial_\alpha F^{\alpha\beta}$$

So that EOM is

$$-J^\beta + \frac{1}{\mu_0}\partial_\alpha F^{\alpha\beta} = 0$$

or

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu}$$

and from Bianchi identity,

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0}$$

In component form:

Use $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_z \\ E_z/c & -B_y & B_z & 0 \end{pmatrix}$$

$F^{ij} = -\epsilon^{ijk}B^k$ $\qquad J^\nu = (c\rho, J_x, J_y, J_z)$

$F^{0i} = -E^i/c$

$F^{i0} = E^i/c$ $\qquad c^2 = \frac{1}{\mu_0\epsilon_0}$

Then $\nu = 0$:

$$\cancel{\partial_0 F^{00}} + \partial_i F^{i0} = \mu_0 J^0$$

$$+\vec{\nabla}\cdot\frac{\vec{E}}{c} = \mu_0 c\rho$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \mu_0 c^2 \rho = \frac{\rho}{\epsilon_0}}$$

And $\nu = j = \{1,2,3\}$

$$\partial_0 F^{0j} + \partial_i F^{ij} = \mu_0 J^j$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{E^j}{c}\right) + \partial_i\left(-\epsilon^{ijk}B^k\right) = \mu_0 J^j$$

$$-\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}}$$

Extra equations: $\tilde{F}^{0i} = -B^i$, $\tilde{F}^{i0} = B^i$, $\tilde{F}^{ij} = +\epsilon^{ijk}\frac{E^k}{c}$

$$\cancel{\partial_0 \tilde{F}^{00}} + \partial_i \tilde{F}^{i0} = 0$$

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

$$\partial_0 \tilde{F}^{0j} + \partial_i \tilde{F}^{ij} = 0$$

$$-\frac{1}{c}\frac{\partial}{\partial t}B^j + \partial_i \epsilon^{ijk}\frac{E^k}{c} = 0$$

$$-\frac{1}{c}\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\times\frac{\vec{E}}{c} = 0$$

$$\boxed{\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$$