Second Noether's theorem for pure Maxwell theory

(can couple to currents)

$$\mathcal{L} = \mathcal{L}(A_\mu, \partial_\nu A_\mu)$$

Consider variations of the type: $A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$ with local gauge parameter

or

$$\delta A_\mu = \frac{-1}{e}\partial_\mu \alpha(x)$$

Then
$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial A_\mu}\delta A_\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\delta(\partial_\nu A_\mu)$$

$$= \frac{\partial \mathcal{L}}{\partial A_\mu}\left(\frac{-1}{e}\partial_\mu \alpha\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\left(\frac{-1}{e}\partial_\nu\partial_\mu\alpha\right) = 0$$

(by gauge invariance)

Use EOM to replace
$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}$$

$$0 = \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right)\left(\frac{-1}{e}\partial_\mu \alpha\right) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\left(\frac{-1}{e}\partial_\nu\partial_\mu\alpha\right) = 0$$

Since $\alpha(x)$ is arbitrary, first derivative indep. of second deriv. $\Rightarrow$ the terms vanish separately.

- Second term implies $\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}$ is antisymmetric $:= -F^{\nu\mu}$
- Then, first term implies $\partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \partial_\nu(-F^{\nu\mu}) = 0$

Therefore, Noether's theorem for gauge transformations is not a conservation law, but an equation of motion.

In fact, these considerations lead us to the requirement that

$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = -F^{\nu\mu}$ is antisymmetric.