Polarization 3-vectors

$\vec{\varepsilon}_\lambda(\hat{k})$ points in direction of $\vec{E}$-field.

– For massless fields, these are "helicity functions", given almost exclusively in cartesian basis. (Quantization axis along $\hat{k}$, angles measured along $\hat{z}$).

They satisfy:
$$\hat{S}^2\,\vec{\varepsilon}_\lambda(\hat{k}) = \hbar\, 1(1+1)\,\vec{\varepsilon}_\lambda(\hat{k}) = 2\hbar\,\varepsilon_\lambda(\hat{k})$$
$$\hat{S}\cdot\hat{k}\;\vec{\varepsilon}_\lambda(\hat{k}) = (\hbar\lambda)\,\vec{\varepsilon}_\lambda(\hat{k})$$

Recall generators of rotations in cartesian basis:

$$\hat{S}_x = \hbar\begin{pmatrix} & & \\ & 0 & -i \\ & i & 0\end{pmatrix} \qquad \hat{S}_y = \hbar\begin{pmatrix}0 & & i\\ & 0 & \\ -i & & 0\end{pmatrix} \qquad \hat{S}_z = \hbar\begin{pmatrix}0 & -i & \\ i & 0 & \\ & & \end{pmatrix}$$

Eigenfunctions of $\hat{S}_z$ are: (Helicity states for motion in $\hat{z}$ dir.)

$$\vec{\varepsilon}_{+1}(\hat{z}) = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\-i\\0\end{pmatrix} \qquad \vec{\varepsilon}_0(\hat{z}) = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \vec{\varepsilon}_{-1}(\hat{z}) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\\0\end{pmatrix}$$

Linear polarization states:

$$\vec{\varepsilon}_x(\hat{z}) = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \vec{\varepsilon}_y(\hat{z}) = \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \vec{\varepsilon}_z(\hat{z}) = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Then, to obtain appropriate polarization vectors for motion along
$\hat{k} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$,

make an active rotation: $\mathbb{R}(\hat{k})\cdot\vec{\varepsilon}(\hat{z})$

$\mathbb{R}(\hat{k})$ = rotation matrix in cartesian basis when acting on helicity states gives Wigner-D.

Helicity (circular) polarization states:

$$\vec{\varepsilon}_{+1}(\hat{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix}-c_\theta c_\phi + i s_\phi\\ -c_\theta s_\phi - i s_\phi\\ s_\theta\end{pmatrix} \qquad \vec{\varepsilon}_0(\hat{k}) = \begin{pmatrix}s_\theta c_\phi\\ s_\theta s_\phi\\ c_\theta\end{pmatrix} = \hat{k} \qquad \vec{\varepsilon}_{-1}(\hat{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix}c_\theta s_\phi - i s_\phi\\ c_\theta s_\phi - i c_\phi\\ -s_\theta\end{pmatrix}$$

LEFT CIRCULAR — LONGITUDINALLY POL. — RIGHT CIRCULAR

Linear polarization states:

$$\vec{\varepsilon}_x(\hat{k}) = \begin{pmatrix}\cos\theta\cos\phi\\ \cos\theta\sin\phi\\ -\sin\theta\end{pmatrix} \equiv \hat{e}_\theta \qquad \vec{\varepsilon}_y(\hat{k}) = \begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix} \equiv \hat{e}_\phi \qquad \vec{\varepsilon}_z(\hat{k}) = \hat{k} \equiv \hat{e}_k$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{-i\hbar}{2} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \left\{ e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} + e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \right\}$$

Write $\vec{k}$'s in front as gradients.

$$= \int \frac{d^3k}{(2\pi)^3} \frac{-i\hbar}{2} \left(\delta_{ij} - \nabla_i \frac{1}{\nabla^2} \nabla_j \right) \left\{ e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} + e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \right\}$$

$$= \left(\delta_{ij} - \nabla_i \frac{1}{\nabla^2} \nabla_j \right) \int \frac{d^3k}{(2\pi)^3} \frac{-i\hbar}{2} \left\{ e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} + \text{c.c.} \right\}$$

$$= \left(\delta_{ij} - \nabla_i \frac{1}{\nabla^2} \nabla_j \right) \left(\frac{-i\hbar}{\cancel{2}} \right) \cancel{2}\, \delta^{(3)}(\vec{x}-\vec{x}')$$

$$= -i\hbar \underbrace{\left(\delta_{ij} - \nabla_i \frac{1}{\nabla^2} \nabla_j \right) \delta^{(3)}(\vec{x}-\vec{x}')}_{\text{transverse delta tensor}} \checkmark$$